

Braid groups and Artin groups

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1 Introduction

The braids go back to several centuries and were universally used for ornamental purposes or even practical ones, for example in the fashioning of ropes. Now, they are described by means of abstract models known under the name of “theory of braids”. The theory of braids studies the concept of braids (such as we imagine them) as well as various generalizations arising from various branches of the mathematics. The idea is that the braids form a group. The number of strands must be fixed so that the operation is well-defined. So, we have a braid group on two strands, a braid group on three strands, and so on. The braid group on one strand is trivial because a string cannot be braided (although it can be knotted).

We generally make the mathematical study of braids go back to an article of Emile Artin [7] dated from 1925, in which is described the notion of braids under various aspects, one being that obvious, like a “series of tended and interlaced strings”, and others more conceptual but equally deep, such as a presentation by generators and relations, or a presentation as the mapping class group of a punctured disk.

Since the 30s, a strong link between braids and links (and knots) were established by people such as Alexander and Markov (see [19]). This link is at the origin in the 80s of a deep revival in the theory of knots with the work of Jones and his invariant defined from the theory of braids (see [101], [102], [83], and [137]).

Later, interesting relations with the algebraic geometry and the theory of finite groups generated by reflections were established, in particular by Arnol’d [5], [3], [6] and Brieskorn [29], [31]. These relations become particularly interesting when we extend the notion of braid groups to that of Artin groups of spherical type, also called generalized braid groups. Although the Artin groups were introduced by Tits [145] as extensions of Coxeter groups, their study really began in the seventies with the works of Brieskorn [30], [31], Saito [32] and Deligne [71], where different aspects of these groups are studied, such as their combinatorics, as well as their link with the hyperplane arrangements and the singularities.

Some problems in group theory, often very close to the algorithmics, such as the word and conjugacy problems, have a renewal of interest not only through their applications in the other domains, but also because the notion of mathematical demonstration is changing. Indeed, we distinguish now the notion of demonstration from the notion of effective demonstration, the one which builds up the solution. Such a demonstration gives rise to an algorithm, and its complexity (calculation time) is of importance. The algorithmics in the braid groups is especially active. Problems of decision such as the conjugacy problem were solved by Garside [85] in 1969 with methods which are now the source of numerous works on the braid groups. In [70] is introduced a

more formal and more general framework to study algorithmic problems on the braid groups: the Garside groups. The idea is to isolate certain combinatorial properties of the braid groups, in particular these emphasized by Garside [85]. It is a less restrictive model which uses tools from the language theory (monoids, rewriting systems) and the combinatorics (ordered sets), tools that are especially adapted to treat algorithmic problems. Now, the major part of the algorithmic problems on the braid groups are studied within the framework of the Garside groups. Also, let us indicate that the Artin groups of spherical type are Garside groups.

This survey is written from these viewpoints but also maintaining two other objectives: (1) to make a survey understandable by non-specialists; (2) to make as often as possible the link with the mapping class groups.

The first section is about the “classical” theory of braid groups. Various aspects as well as some properties of them are presented. The second section is an introduction to the Artin groups, and the third is an introduction to the Garside groups. There, the reader will find algorithms to solve some decision problems such as the conjugacy one for the braid groups (and Garside groups).

The fifth section is about the cohomology of Artin groups, although the exposition goes beyond by explaining the Salvetti complexes. These are tools originally from the theory of hyperplane arrangements that turn out to be useful in the context of the braid groups.

The sixth section is about the linear representations of the braid groups studied by Bigelow [17] and Krammer [105], [106], as well as about its various generalizations (to the Artin groups). Both, the algebraic aspect and the topological aspect of these representations, are explained. Other linear representations of the braid groups have been studied and are also interesting but, for lack of place and for reason of coherence, these will not be treated in this text. We refer to [21] for a survey on the other linear representations.

The seventh section is about the geometric representations of the Artin groups. (By a geometric representation we simply mean a homomorphism in a mapping class group.) This subject is less popular than the previous ones but I strongly believe in its future. In particular, Subsection 7.3, where are explained the results of Castel [40], shows all the power of such a study.

Finally, I would like to indicate two aspects of the braid groups which are not in this survey and which “should be in any survey on the braid groups”.

The first aspect is the link of the braids with links and knots. This is very important in the theory but amply explained in all the books and almost all the surveys on the subject. So, I voluntarily ignore this aspect in order to be able to treat in a more detailed way the other ones. The reader will find in [19], [93], [127], [103] detailed expositions on this aspect and on the braid groups in general.

I would have wanted to make an eighth section to explain the second aspect: the orders in the braid groups. But, unfortunately, this article is long enough

and there is no more room for another section. Inspired by problems of set theory, Dehornoy [65] founded an explicit construction of a total ordering invariant by left multiplication in the braid group. The fact that the braid group is orderable is not maybe completely new, in the sense that it results from Nielsen theory [128], but Dehornoy's ordering is interesting in itself. In my opinion, it is an important tool to understand the braid groups, and I augur numerous developments in this direction. The Artin groups of type B_n and A_n embed into braid groups (see Section 3) thus they are also orderable. The Artin groups of type D_n embed into mapping class groups of surfaces with boundary (see Section 7), and, by [138], such a group is orderable. We do not know whether the other Artin groups are orderable or not. We encourage the reader to consult [68] for a detailed discussion on this subject.

2 Braid groups

2.1 Braids

Let $n \geq 1$ be an integer, and let P_1, \dots, P_n be n distinct points in the plane \mathbb{R}^2 (except mention of the contrary, we will always assume $P_k = (k, 0)$ for all $1 \leq k \leq n$). Define a *braid on n strands* to be a n -tuple $\beta = (b_1, \dots, b_n)$ of paths, $b_k : [0, 1] \rightarrow \mathbb{R}^2$, such that

- $b_k(0) = P_k$ for all $1 \leq k \leq n$;
- there exists a permutation $\chi = \theta(\beta) \in \text{Sym}_n$ such that $b_k(1) = P_{\chi(k)}$ for all $1 \leq k \leq n$;
- $b_k(t) \neq b_l(t)$ for all $k \neq l$ and all $t \in [0, 1]$.

Two braids α and β are said to be *homotopic* if there exists a continuous family $\{\gamma_s\}_{s \in [0, 1]}$ of braids such that $\gamma_0 = \alpha$ and $\gamma_1 = \beta$. Note that $\theta(\alpha) = \theta(\beta)$ if α and β are homotopic.

We represent graphically a homotopy class of braids as follows. Let I_k be a copy of the interval $[0, 1]$. Take a braid $\beta = (b_1, \dots, b_n)$ and define the *geometric braid*

$$\beta^g : I_1 \sqcup \dots \sqcup I_n \rightarrow \mathbb{R} \times [0, 1]$$

by $\beta^g(t) = (b_k(t), t)$ for all $t \in I_k$ and all $1 \leq k \leq n$. Let $\text{proj} : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$ be the projection defined by

$$\text{proj}(x, y, t) = (x, t).$$

Up to homotopy, we can assume that $\text{proj} \circ \beta^g$ is a smooth immersion with only transversal double points that we call *crossings*. In each crossing we indicate graphically like in Figure 2.1 which strand goes over the other. Such

a representation of β is called a *braid diagram* of β . An example is illustrated in Figure 2.2.

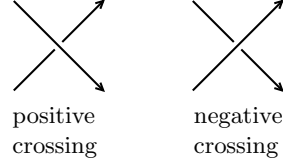


Figure 2.1. Crossings in a braid diagram.

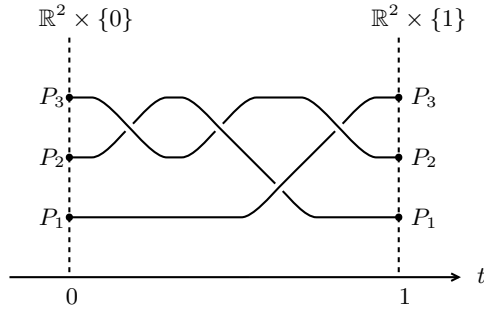


Figure 2.2. A braid diagram.

The *product* of two braids $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ is defined to be the braid

$$\alpha \cdot \beta = (a_1 b_{\chi(1)}, \dots, a_n b_{\chi(n)}),$$

where $\chi = \theta(\alpha)$. An example is illustrated in Figure 2.3.

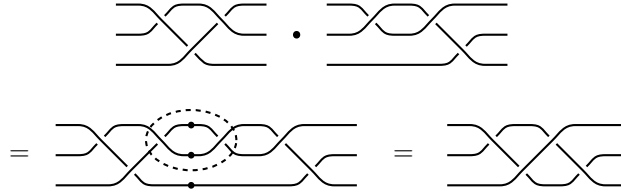


Figure 2.3. Product of two braids.

Let \mathcal{B}_n denote the set of homotopy classes of braids on n strands. It is easily seen that the above defined multiplication of braids induces an operation on \mathcal{B}_n . Moreover, we have the following.

Proposition 2.1. *The set \mathcal{B}_n endowed with this operation is a group.*

From now on, except mention of the contrary, by a braid we will mean a homotopy class of braids. The group \mathcal{B}_n of Proposition 2.1 is called the *braid group on n strands*. The identity is the *constant braid* $\text{Id} = (\text{Id}_1, \dots, \text{Id}_n)$, where, for $1 \leq k \leq n$, Id_k denotes the constant path on P_k . The inverse of a braid β is its *mirror* as illustrated in Figure 2.4.

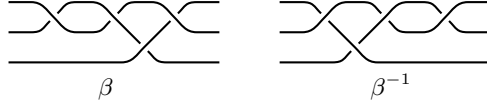


Figure 2.4. Inverse of a braid.

Recall that, if two braids α, α' are homotopic, then $\theta(\alpha) = \theta(\alpha')$. Hence, the map θ from the set of braids on n strands to Sym_n induces a map $\theta : \mathcal{B}_n \rightarrow \text{Sym}_n$. It is easily checked that this map is an epimorphism. Its kernel is called the *pure braid group on n strands* and is denoted by \mathcal{PB}_n . It plays an important role in the theory.

Let σ_k be the braid illustrated in Figure 2.5. One can easily verify that $\sigma_1, \dots, \sigma_{n-1}$ generate the braid group \mathcal{B}_n and satisfy the relations

$$\begin{aligned} \sigma_k \sigma_l &= \sigma_l \sigma_k & \text{if } |k - l| \geq 2, \\ \sigma_k \sigma_l \sigma_k &= \sigma_l \sigma_k \sigma_l & \text{if } |k - l| = 1. \end{aligned}$$

(See Figure 2.6.) These relations suffice to define the braid group, namely:

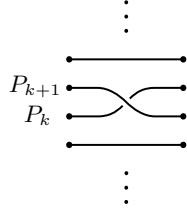


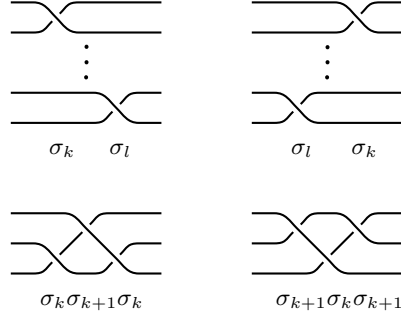
Figure 2.5. The braid σ_k .

Theorem 2.2 (Artin [7], [8], Magnus [118]). *The group \mathcal{B}_n has a presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations*

$$\begin{aligned} \sigma_k \sigma_l &= \sigma_l \sigma_k & \text{if } |k - l| \geq 2, \\ \sigma_k \sigma_l \sigma_k &= \sigma_l \sigma_k \sigma_l & \text{if } |k - l| = 1. \end{aligned}$$

Theorem 2.3 (Burau [35], Markov [123]). *For $1 \leq k < l \leq n$, let*

$$\delta_{kl} = \sigma_{l-1} \cdots \sigma_{k+1} \sigma_k^2 \sigma_{k+1}^{-1} \cdots \sigma_{l-1}^{-1}.$$

**Figure 2.6.** Relations in \mathcal{B}_n .

Then the pure braid group \mathcal{PB}_n has a presentation with generators

$$\delta_{kl}, \quad 1 \leq k < l \leq n,$$

and relations

$$\begin{aligned} \delta_{rs} \delta_{kl} \delta_{rs}^{-1} &= \delta_{kl} && \text{if } 1 \leq r < s < k < l \leq n \\ &&& \text{or } 1 \leq k < r < s < l \leq n, \\ \delta_{rk} \delta_{kl} \delta_{rk}^{-1} &= \delta_{kl}^{-1} \delta_{rl}^{-1} \delta_{kl} \delta_{rl} \delta_{kl} && \text{if } 1 \leq r < k < l \leq n, \\ \delta_{rk} \delta_{rl} \delta_{rk}^{-1} &= \delta_{kl}^{-1} \delta_{rl} \delta_{kl} && \text{if } 1 \leq r < k < l \leq n, \\ \delta_{rs} \delta_{kl} \delta_{rs}^{-1} &= \delta_{sl}^{-1} \delta_{rl}^{-1} \delta_{sl} \delta_{rl} \delta_{kl} \delta_{rl}^{-1} \delta_{sl}^{-1} \delta_{rl} \delta_{sl} && \text{if } 1 \leq r < k < s < l \leq n. \end{aligned}$$

Note. Most of the proofs of Theorems 2.2 and 2.3 that can be found in the literature proceed as follows. Given an exact sequence

$$1 \rightarrow K \longrightarrow G \longrightarrow H \rightarrow 1,$$

there is a machinery to compute a presentation of G from presentations of K and H . We start with the observation that $\mathcal{PB}_2 \simeq \mathbb{Z}$ and with the exact sequence

$$1 \rightarrow F_n \longrightarrow \mathcal{PB}_{n+1} \longrightarrow \mathcal{PB}_n \rightarrow 1, \quad (2.1)$$

where F_n is a free group of rank n , to prove Theorem 2.3 by induction on n . (The exact sequence (2.1) will be explained in Subsection 2.2.) Then we use the exact sequence

$$1 \rightarrow \mathcal{PB}_n \longrightarrow \mathcal{B}_n \longrightarrow \text{Sym}_n \rightarrow 1$$

to prove Theorem 2.2 from Theorem 2.3. Another proof which, as far as I know, is not written in the literature but is known to experts, consists on extracting the presentation of Theorem 2.2 from the Salvetti complex of \mathcal{B}_n . This is a cellular complex which is a $K(\mathcal{B}_n, 1)$ (see Section 5).

2.2 Configuration spaces

We identify \mathbb{R}^2 with \mathbb{C} and P_k with $k \in \mathbb{C}$ for all $1 \leq k \leq n$. For $1 \leq k < l \leq n$ we denote by H_{kl} the linear hyperplane of \mathbb{C}^n defined by the equation $z_k = z_l$. The *big diagonal* of \mathbb{C}^n is defined to be

$$\text{Diag}_n = \bigcup_{1 \leq k < l \leq n} H_{kl}.$$

The *space of ordered configurations of n points* in \mathbb{C} is defined to be

$$M_n = \mathbb{C}^n \setminus \text{Diag}_n.$$

This is the space of n -tuples $\mathbf{z} = (z_1, \dots, z_n)$ of complex numbers such that $z_k \neq z_l$ for $k \neq l$. The symmetric group Sym_n acts freely on M_n . The quotient

$$N_n = M_n / \text{Sym}_n$$

is called the *space of configurations of n points* in \mathbb{C} . This is the space of unordered n -tuples $\mathbf{z} = \{z_1, \dots, z_n\}$ of complex numbers such that $z_k \neq z_l$ for $k \neq l$.

Proposition 2.4. *Let $P_0 = (1, 2, \dots, n) \in M_n$. Then $\pi_1(M_n, P_0) = \mathcal{PB}_n$.*

Proof. For a pure braid $\beta = (b_1, \dots, b_n)$ we set

$$\begin{aligned} \varphi(\beta) : [0, 1] &\rightarrow M_n \\ t &\mapsto (b_1(t), \dots, b_n(t)). \end{aligned}$$

Clearly, $\varphi(\beta)$ is a loop based at P_0 . Moreover, two pure braids α and α' are homotopic if and only if $\varphi(\alpha)$ and $\varphi(\alpha')$ are homotopic. Thus φ induces a bijection $\varphi_* : \mathcal{PB}_n \rightarrow \pi_1(M_n, P_0)$ which turns out to be a homomorphism. \square

For $\mathbf{z} \in M_n$, we denote by $[\mathbf{z}]$ the element of $N_n = M_n / \text{Sym}_n$ represented by \mathbf{z} .

Proposition 2.5. $\pi_1(N_n, [P_0]) = \mathcal{B}_n$.

Proof. For a braid $\beta = (b_1, \dots, b_n)$ we set

$$\begin{aligned} \hat{\varphi}(\beta) : [0, 1] &\rightarrow N_n \\ t &\mapsto [b_1(t), \dots, b_n(t)]. \end{aligned}$$

Clearly, $\hat{\varphi}(\beta)$ is a loop based at $[P_0]$. It is easily checked that $\hat{\varphi}$ induces a homomorphism $\hat{\varphi}_* : \mathcal{B}_n \rightarrow \pi_1(N_n, [P_0])$, and that the following diagram commutes

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{PB}_n & \longrightarrow & \mathcal{B}_n & \longrightarrow & \text{Sym}_n \longrightarrow 1 \\ & & \varphi_* \downarrow \simeq & & \downarrow \hat{\varphi}_* & & \downarrow \text{Id} \\ 1 & \longrightarrow & \pi_1(M_n, P_0) & \longrightarrow & \pi_1(N_n, [P_0]) & \longrightarrow & \text{Sym}_n \longrightarrow 1 \end{array}$$

The first row is exact by definition, and the second one is associated to the regular covering $M_n \rightarrow N_n = M_n/\text{Sym}_n$, so it is exact, too. We conclude by the five lemma that $\hat{\varphi}_*$ is an isomorphism. \square

Let $f, g \in \mathbb{C}[x]$ be two non-constant polynomials. Set

$$\begin{aligned} f &= a_0x^m + a_1x^{m-1} + \cdots + a_m, & a_0 \neq 0, \\ g &= b_0x^n + b_1x^{n-1} + \cdots + b_n, & b_0 \neq 0. \end{aligned}$$

The *Sylvester matrix* of f and g is defined to be

$$\text{Sylv}(f, g) = \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \ddots & \vdots & b_1 & b_0 & \ddots & \vdots \\ \vdots & a_1 & \ddots & 0 & \vdots & b_1 & \ddots & 0 \\ a_m & \vdots & \ddots & a_0 & b_n & \vdots & \ddots & b_0 \\ 0 & a_m & & a_1 & 0 & b_n & & b_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_m & 0 & \cdots & 0 & b_n \end{pmatrix}$$

$\underbrace{\hspace{10em}}_{n \text{ columns}} \quad \underbrace{\hspace{10em}}_{m \text{ columns}}$

The *resultant* of f and g is defined to be

$$\text{Res}(f, g) = \det(\text{Sylv}(f, g)).$$

The following is classical in algebraic geometry (see [54], for example).

Theorem 2.6. *Let $f, g \in \mathbb{C}[x]$ be two non-constant polynomials. Then f and g have a common root if and only if $\text{Res}(f, g) = 0$.*

Corollary 2.7. *Let $f \in \mathbb{C}[x]$ be a polynomial of degree $d \geq 2$. Then f has a multiple root if and only if $\text{Res}(f, f') = 0$.*

The number $\text{Res}(f, f')$ is called the *discriminant* of f and is denoted by $\text{Disc}(f)$. For instance, if $f = ax^2 + bx + c$, then $\text{Disc}(f) = b^2 - 4ac$.

Let $n \geq 2$ and let $\mathbb{C}_n[x]$ be the set of monic polynomials of degree n . In particular, $\mathbb{C}_n[x]$ is isomorphic to \mathbb{C}^n . The map $\text{Disc} : \mathbb{C}_n[x] \rightarrow \mathbb{C}$ is clearly a polynomial function, thus

$$\mathcal{D} = \{f \in \mathbb{C}_n[x]; f \text{ has a multiple root}\} = \{f \in \mathbb{C}_n[x]; \text{Disc}(f) = 0\}$$

is an algebraic hypersurface called the *n -th discriminant*. It is related to the braid group by the following.

Proposition 2.8. $N_n = \mathbb{C}_n[x] \setminus \mathcal{D}$.

Proof. Let $\Phi : M_n \rightarrow \mathbb{C}_n[x] \setminus \mathcal{D}$ be the map defined by

$$\Phi(z_1, \dots, z_n) = (x - z_1) \cdots (x - z_n).$$

Then Φ is surjective and we have $\Phi(\mathbf{u}) = \Phi(\mathbf{v})$ if and only if there exists $\chi \in \text{Sym}_n$ such that $\mathbf{v} = \chi(\mathbf{u})$. Thus $\mathbb{C}_n[x] \setminus \mathcal{D} \simeq M_n / \text{Sym}_n = N_n$. \square

Now, recall the homotopy long exact sequence of a fiber bundle (see [96], for example).

Theorem 2.9. *Let $p : M \rightarrow B$ be a locally trivial fiber bundle. Let $b_0 \in B$, let $F = p^{-1}(b_0)$, and let $P_0 \in F$. Assume that F is connected. Then there is a long exact sequence of homotopy groups*

$$\begin{aligned} \cdots \rightarrow \pi_{k+1}(B, b_0) \rightarrow \pi_k(F, P_0) \rightarrow \pi_k(M, P_0) \rightarrow \pi_k(B, b_0) \rightarrow \cdots \\ \cdots \rightarrow \pi_2(B, b_0) \rightarrow \pi_1(F, P_0) \rightarrow \pi_1(M, P_0) \rightarrow \pi_1(B, b_0) \rightarrow 1. \end{aligned}$$

There are two cases where this long exact sequence becomes a short exact sequence: when $\pi_2(B, b_0) = \{0\}$, and when p admits a cross-section $\kappa : B \rightarrow M$. In the latter case the short exact sequence splits. It turns out that both situations hold in the study of M_n .

Theorem 2.10 (Fadell, Neuwirth [80]). *Let $p : M_{n+1} \rightarrow M_n$ be defined by*

$$p(z_1, \dots, z_n, z_{n+1}) = (z_1, \dots, z_n).$$

Then p is a locally trivial fiber bundle which admits a cross-section $\kappa : M_n \rightarrow M_{n+1}$.

Let $b_0 = (1, 2, \dots, n)$. Then the fiber $p^{-1}(b_0)$ is naturally homeomorphic to $\mathbb{C} \setminus \{1, 2, \dots, n\}$ whose fundamental group is the free group F_n of rank n . A cross-section of p is the map $\kappa : M_n \rightarrow M_{n+1}$ defined by

$$\kappa(z_1, \dots, z_n) = (z_1, \dots, z_n, |z_1| + \cdots + |z_n| + 1).$$

Corollary 2.11. *Let $n \geq 2$. Then there is a split exact sequence*

$$1 \longrightarrow F_n \longrightarrow \mathcal{PB}_{n+1} \begin{array}{c} \xrightarrow{p_*} \\ \xleftarrow{\kappa_*} \end{array} \mathcal{PB}_n \longrightarrow 1.$$

A connected CW-complex X is called $K(\pi, 1)$ if its universal cover is contractible. Equivalently, X is $K(\pi, 1)$ if $\pi_k(X) = \{0\}$ for all $k \geq 2$. In particular, a space X is $K(\pi, 1)$ if and only if some of its connected cover Y is $K(\pi, 1)$. The notion of $K(\pi, 1)$ spaces is of importance in the calculation of the (co)homology of groups. We refer to [34] for detailed explanations on the subject.

It is easily seen that $\mathbb{C} \setminus \{1, \dots, n\}$ is $K(\pi, 1)$, thus, from Theorems 2.9 and 2.10 follows:

Corollary 2.12. *The spaces M_n and N_n are $K(\pi, 1)$.*

It is also known that the fundamental group of a finite dimensional $K(\pi, 1)$ space is torsion free (see [34]), thus:

Corollary 2.13. *$\mathcal{B}_n = \pi_1(N_n)$ is torsion free.*

2.3 Mapping class groups

Let Σ be an oriented compact surface, possibly with boundary, and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a collection of n punctures in the interior of Σ . Let $\text{Homeo}^+(\Sigma, \mathcal{P})$ denote the group of homeomorphisms $h : \Sigma \rightarrow \Sigma$ which preserve the orientation, which pointwise fix the boundary of Σ , and such that $h(\mathcal{P}) = \mathcal{P}$. Let $\text{Homeo}_0^+(\Sigma, \mathcal{P})$ denote the connected component of the identity in $\text{Homeo}(\Sigma, \mathcal{P})$. The *mapping class group* of the pair (Σ, \mathcal{P}) is defined to be

$$\mathcal{M}(\Sigma, \mathcal{P}) = \pi_0(\text{Homeo}^+(\Sigma, \mathcal{P})) = \text{Homeo}^+(\Sigma, \mathcal{P}) / \text{Homeo}_0^+(\Sigma, \mathcal{P}).$$

A *braid* of Σ based at \mathcal{P} is defined to be a n -tuple $\beta = (b_1, \dots, b_n)$ of paths, $b_k : [0, 1] \rightarrow \Sigma$, such that

- $b_k(0) = P_k$ for all $1 \leq k \leq n$;
- there exists a permutation $\chi = \theta(\beta) \in \text{Sym}_n$ such that $b_k(1) = P_{\chi(k)}$ for all $1 \leq k \leq n$;
- $b_k(t) \neq b_l(t)$ for all $k \neq l$ and all $t \in [0, 1]$.

The homotopy classes of braids based at \mathcal{P} form a group denote by $\mathcal{B}_n(\Sigma, \mathcal{P})$ and called the *braid group of Σ on n strands based at \mathcal{P}* . It does not depend up to isomorphism on the choice of \mathcal{P} but only on the cardinality $n = |\mathcal{P}|$. So, we may often write $\mathcal{B}_n(\Sigma)$ in place of $\mathcal{B}_n(\Sigma, \mathcal{P})$. If $\Sigma = \mathbb{D}$ is a disk, then $\mathcal{B}_n(\Sigma)$ is naturally isomorphic to the braid group \mathcal{B}_n .

For $1 \leq k < l \leq n$, we denote by $H_{kl}(\Sigma)$ the set of n -tuples $\mathbf{x} = (x_1, \dots, x_n) \in \Sigma^n$ such that $x_k = x_l$. The *big diagonal* of Σ^n is defined to be

$$\text{Diag}_n(\Sigma) = \bigcup_{1 \leq k < l \leq n} H_{kl}(\Sigma).$$

The *space of ordered configurations of n points in Σ* is defined to be

$$M_n(\Sigma) = \Sigma^n \setminus \text{Diag}_n(\Sigma).$$

This is the space of n -tuples $\mathbf{x} = (x_1, \dots, x_n)$ in Σ^n such that $x_k \neq x_l$ for all $1 \leq k \neq l \leq n$. The symmetric group Sym_n acts freely on $M_n(\Sigma)$, and the quotient

$$N_n(\Sigma) = M_n(\Sigma) / \text{Sym}_n$$

is called the *space of configurations of n points in Σ* . This is the space of unordered n -tuples $\mathbf{x} = \{x_1, \dots, x_n\}$ of elements of Σ such that $x_k \neq x_l$ for all $1 \leq k \neq l \leq n$.

Set $\mathbf{P}_0 = (P_1, \dots, P_n) \in M_n(\Sigma)$. For $\mathbf{x} \in M_n(\Sigma)$, we denote by $[\mathbf{x}]$ the element of $N_n(\Sigma)$ represented by \mathbf{x} . The following can be proved in the same way as Proposition 2.5.

Proposition 2.14. $\pi_1(N_n(\Sigma), [\mathbf{P}_0]) \simeq \mathcal{B}_n(\Sigma)$.

Now, the surface braid groups and the mapping class groups are related by the following exact sequence.

Theorem 2.15 (Birman [18]). *Suppose Σ is neither a sphere, nor a torus. Then we have the exact sequence*

$$1 \rightarrow \mathcal{B}_n(\Sigma, \mathcal{P}) \longrightarrow \mathcal{M}(\Sigma, \mathcal{P}) \longrightarrow \mathcal{M}(\Sigma) \rightarrow 1.$$

Note. Let

$$\begin{array}{ccc} \Phi : \text{Homeo}^+(\Sigma) & \rightarrow & N_n(\Sigma) \\ \varphi & \mapsto & \{\varphi(P_1), \dots, \varphi(P_n)\}. \end{array}$$

Then Φ is a locally trivial fiber bundle, and the fiber of Φ over $\mathcal{P} = [\mathbf{P}_0]$ is $\text{Homeo}^+(\Sigma, \mathcal{P})$. Furthermore, it is known that $\pi_1(\text{Homeo}^+(\Sigma)) = \{1\}$ (see [92]), thus, by the homotopy long exact sequence of a fiber bundle (see [96]), we have the short exact sequence

$$1 \rightarrow \pi_1(N_n(\Sigma), \mathcal{P}) \longrightarrow \pi_0(\text{Homeo}^+(\Sigma, \mathcal{P})) \longrightarrow \pi_0(\text{Homeo}^+(\Sigma)) \rightarrow 1,$$

which is the same as the exact sequence of Theorem 2.15.

It is known that $\mathcal{M}(\mathbb{D}) = \{1\}$ (see [1]), thus, by Theorem 2.15:

Theorem 2.16 (Artin [7], [8]). *Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a collection of n punctures in the interior of the disk \mathbb{D} . Then $\mathcal{M}(\mathbb{D}, \mathcal{P}) \simeq \mathcal{B}_n$.*

The isomorphism $\Phi : \mathcal{M}(\mathbb{D}, \mathcal{P}) \rightarrow \mathcal{B}_n$ can be easily described as follows. Let $\varphi \in \text{Homeo}^+(\mathbb{D}, \mathcal{P})$. We know by [1] that $\pi_0(\text{Homeo}^+(\mathbb{D})) = \{1\}$, thus there exists a continuous path $\{\varphi_t\}_{t \in [0,1]}$ in $\text{Homeo}^+(\mathbb{D})$ such that $\varphi_0 = \text{Id}$ and $\varphi_1 = \varphi$. Let $\beta = (b_1, \dots, b_n)$ be the braid defined by

$$b_k(t) = \varphi_t(P_k), \quad 1 \leq k \leq n \text{ and } t \in [0, 1].$$

Then $\Phi(\varphi)$ is the homotopy class of β .

The reverse isomorphism $\Phi^{-1} : \mathcal{B}_n \rightarrow \mathcal{M}(\mathbb{D}, \mathcal{P})$ is more complicated to describe, but the images of the standard generators can be easily defined in terms of braid twists as follows.

We come back to the situation where Σ is an oriented compact surface and $\mathcal{P} = \{P_1, \dots, P_n\}$ is a collection of n punctures in the interior of Σ . Let $P_k, P_l \in \mathcal{P}$, $k \neq l$. An *essential arc* joining P_k to P_l is defined to be an embedding $a : [0, 1] \rightarrow \Sigma$ such that $a(0) = P_k$, $a(1) = P_l$, $a((0, 1)) \cap \mathcal{P} = \emptyset$, and $a([0, 1]) \cap \partial\Sigma = \emptyset$. Two essential arcs a and a' are said to be *isotopic* if there is a continuous family $\{a_t\}_{t \in [0, 1]}$ of essential arcs such that $a_0 = a$ and $a_1 = a'$. Isotopy of essential arcs is an equivalence relation that we denote by $a \sim a'$.

Let a be an essential arc joining P_k to P_l . Let $\mathbb{D} = \{z \in \mathbb{C}; |z| \leq 1\}$ be the standard disk, and let $A : \mathbb{D} \rightarrow \Sigma$ be an embedding such that

- $a(t) = A(t - \frac{1}{2})$ for all $t \in [0, 1]$;
- $A(\mathbb{D}) \cap \mathcal{P} = \{P_k, P_l\}$.

Let $T \in \text{Homeo}^+(\Sigma, \mathcal{P})$ be defined by

$$(T \circ A)(z) = A(e^{2i\pi|z|}z), \quad z \in \mathbb{D},$$

and T is the identity outside the image of A (see Figure 2.7). The *braid twist along a* is defined to be the element $\tau_a \in \mathcal{M}(\Sigma, \mathcal{P})$ represented by T , that is, the isotopy class of T . Note that:

- the definition of τ_a does not depend on the choice of $A : \mathbb{D} \rightarrow \Sigma$;
- if a is isotopic to a' , then $\tau_a = \tau_{a'}$.

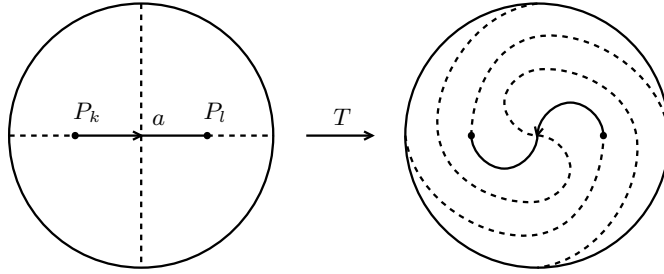


Figure 2.7. Braid twist.

Now, we view the disk \mathbb{D} as the disk in \mathbb{C} of radius $\frac{n+1}{2}$ centered at $\frac{n+1}{2}$, and we set $P_k = k$ for $1 \leq k \leq n$. Let $a_k : [0, 1] \rightarrow \mathbb{D}$ be the arc defined by

$$a_k(t) = k + t, \quad t \in [0, 1].$$

(See Figure 2.8.) Then:

Lemma 2.17. *The reverse isomorphism $\Phi^{-1} : \mathcal{B}_n \rightarrow \mathcal{M}(\mathbb{D}, \mathcal{P})$ is defined by*

$$\Phi^{-1}(\sigma_k) = \tau_{a_k}, \quad 1 \leq k \leq n-1.$$

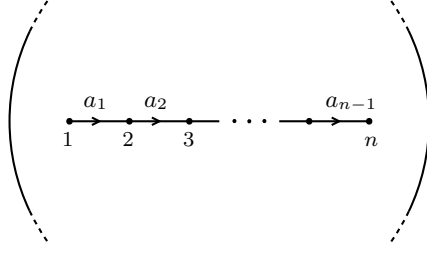


Figure 2.8. The standard generators of $\mathcal{M}(\mathbb{D}, \mathcal{P}) = \mathcal{B}_n$.

2.4 Automorphisms of free groups

For a group G , we denote by $\text{Aut}(G)$ the group of automorphisms of G , by $\text{Inn}(G)$ the group of inner automorphisms of G , and by $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$ the group of outer automorphisms of G .

Let $F_n = F(x_1, \dots, x_n)$ be the free group of rank n . For $1 \leq k \leq n-1$, let $\tau_k : F_n \rightarrow F_n$ be the automorphism defined by

$$\tau_k : \begin{cases} x_k & \mapsto x_k^{-1} x_{k+1} x_k \\ x_{k+1} & \mapsto x_k \\ x_l & \mapsto x_l \end{cases} \quad \text{if } l \neq k, k+1$$

One can easily show the following.

Proposition 2.18. *The mapping $\sigma_k \mapsto \tau_k$, $1 \leq k \leq n-1$, determines a representation $\rho : \mathcal{B}_n \rightarrow \text{Aut}(F_n)$.*

The above representation $\rho : \mathcal{B}_n \rightarrow \text{Aut}(F_n)$ is called the *Artin representation*. It is faithful, more precisely:

Theorem 2.19 (Artin [7], [8]). (1) *The Artin representation $\rho : \mathcal{B}_n \rightarrow \text{Aut}(F_n)$ is faithful.*

(2) *An automorphism $\alpha \in \text{Aut}(F_n)$ belongs to $\text{Im } \rho$ if and only if $\alpha(x_n \cdots x_2 x_1) = x_n \cdots x_2 x_1$ and there exists a permutation $\chi \in \text{Sym}_n$ such that $\alpha(x_k)$ is conjugate to $x_{\chi(k)}$ for all $1 \leq k \leq n$.*

In particular, \mathcal{B}_n can be viewed as a subgroup of $\text{Aut}(F_n)$. This has some consequences on \mathcal{B}_n itself such as the two properties defined below.

A group G is called *residually finite* if for all $g \in G \setminus \{1\}$ there exists a homomorphism $\varphi : G \rightarrow H$ such that H is finite and $\varphi(g) \neq 1$. A group G is called *Hopfian* if every epimorphism $\varphi : G \rightarrow G$ is an isomorphism. It is known that the subgroups of $\text{Aut}(F_n)$ are both, residually finite and Hopfian (see [119]), thus, by Theorem 2.19:

Corollary 2.20. *The braid group \mathcal{B}_n is residually finite and Hopfian.*

There are several ways to describe geometrically the Artin representation. The first way is using the Fadell-Neuwirth fiber bundle $p : M_{n+1} \rightarrow M_n$ of Theorem 2.10. Let Sym_n act on M_n and on M_{n+1} . The second action is on the first n coordinates, that is,

$$\chi(z_1, \dots, z_n, z_{n+1}) = (z_{\chi^{-1}(1)}, \dots, z_{\chi^{-1}(n)}, z_{n+1}), \quad \text{for } \chi \in \text{Sym}_n.$$

The map $p : M_{n+1} \rightarrow M_n$ induces a map $\bar{p} : M_{n+1}/\text{Sym}_n \rightarrow M_n/\text{Sym}_n = N_n$ which turns out to be a locally trivial fiber bundle. The fiber is again homeomorphic to $\mathbb{C} \setminus \{1, 2, \dots, n\}$, and $\bar{p} : M_{n+1}/\text{Sym}_n \rightarrow N_n$ has also a cross-section $\bar{\kappa} : N_n \rightarrow M_{n+1}/\text{Sym}_n$. So, from the homotopy long exact sequence of a fiber bundle (see Theorem 2.9) we obtain the following split exact sequence

$$1 \rightarrow F_n = \pi_1(\mathbb{C} \setminus \{1, \dots, n\}) \longrightarrow \pi_1(M_{n+1}/\text{Sym}_n) \xrightarrow[\bar{\kappa}_*]{\bar{p}_*} \pi_1(N_n) = \mathcal{B}_n \rightarrow 1. \quad (2.2)$$

The action of $\mathcal{B}_n = \pi_1(N_n)$ on $F_n = \pi_1(\mathbb{C} \setminus \{1, \dots, n\})$ derived from the above split exact sequence is exactly the Artin representation.

Another way to represent the Artin representation is using the isomorphism $\mathcal{B}_n \simeq \mathcal{M}(\mathbb{D}, \{P_1, \dots, P_n\})$. Fix a basepoint $P_0 \in \partial\mathbb{D}$. Then it is easily shown that $\mathcal{M}(\mathbb{D}, \{P_1, \dots, P_n\})$ acts on $\pi_1(\mathbb{D} \setminus \{P_1, \dots, P_n\}, P_0) = F_n$, and that this action is the Artin representation.

The latter point of view of the Artin representations can be extended to all the mapping class groups. In this setting, it is known as the Dehn-Nielsen-Baer theorem. Here is a version of this theorem.

Theorem 2.21 (Dehn, Nielsen [128], Baer [9], Magnus [118]). *Let Σ be a closed oriented surface, and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a collection of n punctures in Σ . Then the natural homomorphism $\rho : \mathcal{M}(\Sigma, \mathcal{P}) \rightarrow \text{Out}(\pi_1(\Sigma \setminus \mathcal{P}))$ is injective. Moreover, if $\mathcal{P} = \emptyset$, then the image of ρ is an index 2 subgroup of $\text{Out}(\pi_1(\Sigma))$.*

We refer to [99] for a detailed exposition on the Dehn-Nielsen-Baer theorem which include other versions of it.

Note. There are some variants of the Artin representations introduced in [150] and [59] that lead to invariants of links.

3 Artin groups

3.1 Definitions and examples

Let S be a finite set. A *Coxeter matrix* over S is a square matrix $M = (m_{st})_{s,t \in S}$ indexed by the elements of S such that

- $m_{ss} = 1$ for all $s \in S$;
- $m_{st} = m_{ts} \in \{2, 3, 4, \dots, +\infty\}$ for all $s, t \in S$, $s \neq t$.

A Coxeter matrix $M = (m_{st})_{s,t \in S}$ is usually represented by its *Coxeter graph*, $\Gamma = \Gamma(M)$. This is a labeled graph defined by the following data.

- S is the set of vertices of Γ .
- Two vertices $s, t \in S$, $s \neq t$, are joined by an edge if $m_{st} \geq 3$. This edge is labeled by m_{st} if $m_{st} \geq 4$.

Let Γ be a Coxeter graph. Define the *Coxeter system of type Γ* to be the pair (W, S) , where $W = W_\Gamma$ is the group presented by the generating set S and the relations

$$\begin{aligned} s^2 &= 1 && \text{for all } s \in S, \\ (st)^{m_{st}} &= 1 && \text{for all } s, t \in S, s \neq t, \text{ and } m_{st} \neq +\infty, \end{aligned}$$

where $M = (m_{st})_{s,t \in S}$ is the Coxeter matrix of Γ . The group $W = W_\Gamma$ is called the *Coxeter group* of type Γ .

If a, b are two letters and $m \in \mathbb{N}$, then $\text{prod}(a, b : m)$ denotes the word

$$\text{prod}(a, b : m) = \begin{cases} (ab)^{\frac{m}{2}} & \text{if } m \text{ is even,} \\ (ab)^{\frac{m-1}{2}}a & \text{if } m \text{ is odd.} \end{cases}$$

Let $\Sigma = \{\sigma_s; s \in S\}$ be an abstract set in one-to-one correspondence with S . Define the *Artin system of type Γ* to be the pair (G, Σ) , where $G = G_\Gamma$ is the group presented by the generating set Σ and the relations

$$\text{prod}(\sigma_s, \sigma_t : m_{st}) = \text{prod}(\sigma_t, \sigma_s : m_{st}) \quad \text{for } s, t \in S, s \neq t, \text{ and } m_{st} \neq +\infty.$$

The group G is called the *Artin group of type Γ* .

It is easily checked that the group W_Γ is also presented by the generating set S and the relations

$$\begin{aligned} s^2 &= 1 && \text{for all } s \in S, \\ \text{prod}(s, t : m_{st}) &= \text{prod}(t, s : m_{st}) && \text{for all } s, t \in S, s \neq t \text{ and } m_{st} \neq +\infty. \end{aligned}$$

This shows that the mapping $\Sigma \rightarrow S$, $\sigma_s \mapsto s$, induces a *canonical epimorphism* $\theta : G_\Gamma \rightarrow W_\Gamma$.

If $m_{st} = 2$, then

$$\sigma_s \sigma_t = \text{prod}(\sigma_s, \sigma_t : m_{st}) = \text{prod}(\sigma_t, \sigma_s : m_{st}) = \sigma_t \sigma_s,$$

that is, σ_s and σ_t commute. So, if $\Gamma_1, \dots, \Gamma_l$ are the connected components of Γ , then

$$G_\Gamma = G_{\Gamma_1} \times G_{\Gamma_2} \times \cdots \times G_{\Gamma_l}.$$

Similarly, we have

$$W_\Gamma = W_{\Gamma_1} \times W_{\Gamma_2} \times \cdots \times W_{\Gamma_l}.$$

We say that G_Γ (or W_Γ) is *irreducible* if Γ is connected. We say that Γ (or G_Γ) is of *spherical type* if W_Γ is finite.

Example 1. Suppose that Γ is the graph A_n of Figure 3.1. Then $W_\Gamma = \text{Sym}_{n+1}$ is the symmetric group of $\{1, \dots, n, n+1\}$, and the Coxeter generators are the transpositions $s_1 = (1, 2), s_2 = (2, 3), \dots, s_n = (n, n+1)$. The Artin group G_Γ is the braid group \mathcal{B}_{n+1} on $n+1$ strands, and the Artin generators are the standard generators of \mathcal{B}_{n+1} given in Theorem 2.2. The canonical epimorphism coincides with the epimorphism described in Subsection 2.1.

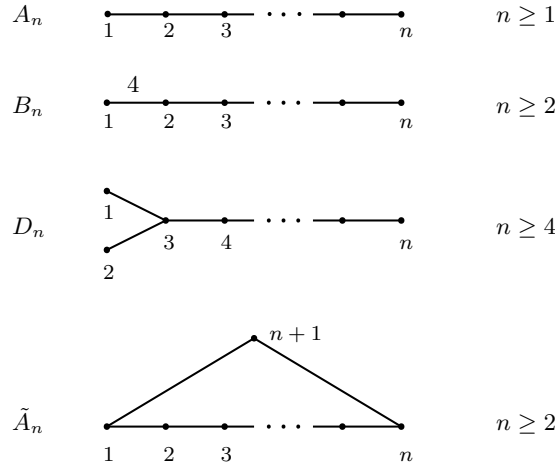


Figure 3.1. The Coxeter graphs A_n , B_n , D_n , and \tilde{A}_n .

Example 2. Suppose that Γ is the Coxeter graph B_n of Figure 3.1. Let $C_2 = \{\pm 1\}$ denote the cyclic group of order 2. Set $\text{Cub}_n = C_2^n \rtimes \text{Sym}_n$, where Sym_n acts on C_2^n by permutation of the coordinates. This is the group of isometries of a regular n -cube (see [94], for example). The group Cub_n is the Coxeter group of type B_n , and the Coxeter generators are

$$s_1 = (-1, 1, \dots, 1) \in C_2^n, \quad s_i = (i-1, i) \in \text{Sym}_n \text{ for } 2 \leq i \leq n.$$

Recall the Artin representation $\rho : \mathcal{B}_n \rightarrow \text{Aut}(F_n)$ defined in Subsection 2.4. Set $G = F_n \rtimes_{\rho} \mathcal{B}_n$. Recall also the action of Sym_n on M_{n+1} defined in Subsection 2.4. It follows from the exact sequence (2.2) that $G = \pi_1(M_{n+1}/\text{Sym}_n)$. In particular, G is an index $n+1$ subgroup of $\pi_1(M_{n+1}/\text{Sym}_{n+1}) = \pi_1(N_{n+1}) = \mathcal{B}_{n+1} = G_{A_n}$. Now, G is the Artin group of type B_n , and the Artin generators are

$$\tau_1 = x_1 \in F_n, \quad \tau_i = \sigma_{i-1} \in \mathcal{B}_n \text{ for } 2 \leq i \leq n.$$

(See [60]).

Example 3. Suppose that Γ is the Coxeter graph D_n of Figure 3.1, where $n \geq 4$. Let $\text{sgn} : C_2^n \rightarrow C_2$ be the homomorphism defined by

$$\text{sgn}(\varepsilon_1, \dots, \varepsilon_n) = \prod_{i=1}^n \varepsilon_i,$$

and let K be the kernel of sgn . The subgroup K is invariant under the action of Sym_n , thus one can consider the subgroup $W = K \rtimes \text{Sym}_n$ of $\text{Cub}_n = C_2^n \rtimes \text{Sym}_n$. This is the Coxeter group of type D_n , and the Coxeter generators are

$$s_1 = (-1, -1, 1, \dots, 1) \cdot (1, 2), \quad s_i = (1, 1, 1, \dots, 1) \cdot (i-1, i) \text{ for } 2 \leq i \leq n.$$

(See [94], for example).

Let $F_{n-1} = F(y_1, \dots, y_{n-1})$ be a free group of rank $n-1$. Let $\rho_{D,1} : F_{n-1} \rightarrow F_{n-1}$ be the automorphism defined by

$$\rho_{D,1} : \begin{cases} y_1 & \mapsto y_1, \\ y_j & \mapsto y_1^{-1} y_j \end{cases} \quad \text{if } j \geq 2.$$

For $2 \leq i \leq n-1$, let $\rho_{D,i} : F_{n-1} \rightarrow F_{n-1}$ be the automorphism defined by

$$\rho_{D,i} : \begin{cases} y_{i-1} & \mapsto y_i, \\ y_i & \mapsto y_i y_{i-1}^{-1} y_i, \\ y_j & \mapsto y_j \end{cases} \quad \text{if } j \neq i-1, i.$$

One can easily show the following.

Lemma 3.1. *The mapping $\sigma_i \mapsto \rho_{D,i}$, $1 \leq i \leq n-1$, determines a representation $\rho_D : \mathcal{B}_n \rightarrow \text{Aut}(F_{n-1})$.*

The following is implicit in [134] and explicit in [60].

Theorem 3.2 (Perron, Vannier [134]). *The representation $\rho_D : \mathcal{B}_n \rightarrow \text{Aut}(F_{n-1})$ is faithful, and the semidirect product $F_{n-1} \rtimes_{\rho_D} \mathcal{B}_n$ is isomorphic to the Artin group G_{D_n} of type D_n .*

Note. It was shown by Allcock [2] that the Artin group G_{D_n} of type D_n can be also presented as an index 2 subgroup of the n -strand braid group of a plane with a single orbifold point of degree 2.

Example 4. Suppose that Γ is the graph \tilde{A}_n of Figure 3.1. Let Sym_{n+1} act on \mathbb{Z}^{n+1} by permutations of the coordinates. Then $\mathbb{Z}^{n+1} \rtimes \text{Sym}_{n+1}$ is the Coxeter group of type Γ (see [28]).

Let $\Phi : G_{B_{n+1}} \rightarrow \mathbb{Z}$ be the homomorphism defined by

$$\Phi(\sigma_1) = 1, \quad \Phi(\sigma_i) = 0 \text{ for } 2 \leq i \leq n.$$

It was observed by several authors [2], [47], [73], [104], that the kernel of Φ is isomorphic to the Artin group $G_{\tilde{A}_n}$ of type \tilde{A}_n . In particular, $G_{\tilde{A}_n}$ is a subgroup of \mathcal{B}_{n+2} .

The Artin generators of $G_{\tilde{A}_n}$, viewed as a subgroup of $\mathcal{B}_{n+2} = \mathcal{M}(\mathbb{D}, \{P_1, P_2, \dots, P_{n+2}\})$, can be described in terms of braid twists as follows. We place P_1, \dots, P_{n+2} in the interior of \mathbb{D} like in Figure 3.2. For $1 \leq i \leq n+1$, let τ_i denote the braid twist along the arc a_i . Then $\tau_1, \dots, \tau_{n+1}$ are the Artin generators of $G_{\tilde{A}_n}$.

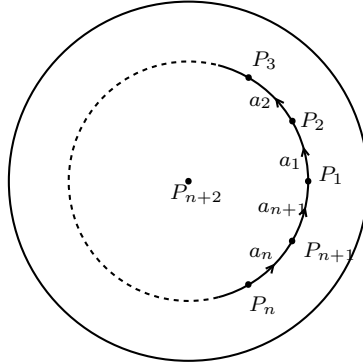


Figure 3.2. Standard generators of $G_{\tilde{A}_n}$.

Note. For a group G we denote by $Z(G)$ the center of G . If $\Gamma = A_n, B_n$, or \tilde{A}_n , then $G_\Gamma/Z(G_\Gamma)$ can be viewed as a finite index subgroup of the mapping class group of a punctured sphere. This has been cleverly exploited to study the group G_Γ itself, in particular, to compute the group of automorphisms of G_Γ (see [44], [10]). Note that the center of G_{A_n} and G_{B_n} is an infinite cyclic group (see [71], [32]), and the center of $G_{\tilde{A}_n}$ is trivial (see [100]).

3.2 Coxeter groups

The Coxeter groups were introduced by Tits [147] in a manuscript which was recently published, and whose results appeared in the seminal Bourbaki's book [28]. The present subsection is a brief survey on these groups with a special emphasis on the results that are needed to study the Artin groups. Standard references for the subject are [28], [97].

Let Γ be a Coxeter graph, let $M = (m_{st})_{s,t \in S}$ be its associated Coxeter matrix, and let (W, S) be the Coxeter system of type Γ .

Let $\Pi = \{e_s; s \in S\}$ be an abstract set in one-to-one correspondence with S , whose elements are called *simple roots*. We denote by V the real vector space having Π as a basis, and by $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ the symmetric bilinear form defined by

$$\langle e_s, e_t \rangle = \begin{cases} -\cos(\frac{\pi}{m_{st}}) & \text{if } m_{st} \neq +\infty, \\ -1 & \text{if } m_{st} = +\infty. \end{cases}$$

For $s \in S$ we define the reflection $r_s : V \rightarrow V$ by

$$r_s(\mathbf{x}) = \mathbf{x} - 2\langle \mathbf{x}, e_s \rangle e_s, \quad \mathbf{x} \in V.$$

Theorem 3.3 (Tits [147]). *The mapping $s \mapsto r_s$, $s \in S$, determines a faithful linear representation $\rho : W \rightarrow GL(V)$.*

The above linear representation is called the *canonical representation* of (W, S) . Note that the bilinear form $\langle \cdot, \cdot \rangle$ is invariant under the action of W .

The *root system* Φ of (W, S) is defined to be the orbit of Π under the action of W , that is,

$$\Phi = \{w \cdot e_s; w \in W, s \in S\}.$$

Let $f \in \Phi$. Write $f = \sum_{s \in S} \lambda_s e_s$, where $\lambda_s \in \mathbb{R}$ for all $s \in S$. We say that f is a *positive root* (resp. a *negative root*) if $\lambda_s \geq 0$ (resp. $\lambda_s \leq 0$) for all $s \in S$. The set of positive roots (resp. negative roots) is denoted by Φ_+ (resp. by Φ_-). The following is proved in [28] for the finite root systems, but the same proof works in general (see also [97], [72]).

Proposition 3.4. *We have the disjoint union $\Phi = \Phi_+ \sqcup \Phi_-$.*

Let A be a finite set that we call an *alphabet*. Let A^* denote the set of finite sequences of elements of A that we call *words on A* . We define an operation on A^* by

$$(a_1, \dots, a_p) \cdot (b_1, \dots, b_q) = (a_1, \dots, a_p, b_1, \dots, b_q).$$

Clearly, A^* endowed with this operation is a monoid which is called the *free monoid* on A . The unit in A^* is the empty word $\epsilon = ()$.

Each element w in the Coxeter group W can be written in the form $w = s_1 s_2 \cdots s_l$, where $s_1, s_2, \dots, s_l \in S$. If l is as small as possible, then l is called the *word length* of w and is denoted by $l = \lg_S(w)$. If $w = s_1 s_2 \cdots s_l$, then the word $\omega = (s_1, s_2, \dots, s_l)$ is called an *expression* of w . If in addition $l = \lg_S(w)$, then ω is called a *reduced expression* of w .

For $w \in W$ we set

$$\Phi_w = \{f \in \Phi_+; w^{-1}f \in \Phi_-\}.$$

Then the word length and the root systems are related by the following.

Proposition 3.5 (Bourbaki [28]). *We have $|\Phi_w| = \lg_S(w)$ for all $w \in W$.*

Let G be a group. A subset $S \subset G$ is called a *positive generating set* of G if it generates G as a monoid. Let S be a positive generating set of G . for $\omega \in S^*$, we denote by $\bar{\omega}$ the element of G represented by ω . A *solution to the word problem* for G is an algorithm which, given $\omega \in S^*$, decides whether $\bar{\omega}$ is trivial or not.

We turn now to describe Tits' solution to the word problem for Coxeter groups.

Let $\omega, \omega' \in S^*$. We say that ω is *transformable to ω' by an M -operation of type I* if there exist $\omega_1, \omega_2 \in S^*$ and $s \in S$ such that

$$\omega = \omega_1 \cdot (s, s) \cdot \omega_2 \quad \text{and} \quad \omega' = \omega_1 \cdot \omega_2.$$

We say that ω is *transformable to ω' by an M -operation of type II* if there exist $\omega_1, \omega_2 \in S^*$ and $s, t \in S$ such that $s \neq t$, $m_{st} \neq +\infty$,

$$\omega = \omega_1 \cdot \text{prod}(s, t : m_{st}) \cdot \omega_2 \quad \text{and} \quad \omega' = \omega_1 \cdot \text{prod}(t, s : m_{st}) \cdot \omega_2.$$

Note that an M -operation of type I shortens the length of the word, but not an M -operation of type II. An M -operation of type II is reversible, but not an M -operation of type I. If ω is transformable to ω' by an M -operation, then $\bar{\omega} = \bar{\omega}'$.

A word ω is called *M -reduced* if its length cannot be reduced by means of M -operations.

Theorem 3.6 (Tits [146]). (1) *A word $\omega \in S^*$ is reduced if and only if it is M -reduced.*

(2) *Let $\omega, \omega' \in S^*$ be two reduced words. We have $\bar{\omega} = \bar{\omega}'$ if and only if one can pass from ω to ω' with a finite sequence of M -operations of type II.*

Now, we introduce a partial order on the Coxeter group W whose role is of importance in the study of the associated Artin group and monoid.

For $u, v \in W$, we set $u \leq_L v$ if there exists $w \in W$ such that $v = uw$ and $\lg_S(v) = \lg_S(u) + \lg_S(w)$.

- Proposition 3.7** (Bourbaki [28]). (1) *Let $u, v \in W$. There exists a unique $w^o \in W$ such that $w^o \leq_L u$, $w^o \leq_L v$, and $w \leq_L w^o$ whenever $w \leq_L u$ and $w \leq_L v$.*
- (2) *Suppose that W is finite. Let $u, v \in W$. There exists a unique $w_o \in W$ such that $u \leq_L w_o$, $v \leq_L w_o$, and $w_o \leq_L w$ whenever $u \leq_L w$ and $v \leq_L w$.*

The element w^o of Proposition 3.7 is denoted by $w^o = u \wedge_L v$, and the element w_o is denoted by $w_o = u \vee_L v$ (if it exists). Note that, by the above, (W, \leq_L) is a lattice if W is finite. In that case, W has a greatest element which is often denoted by w_0 .

We finish the subsection with the classification of the spherical type Coxeter graphs.

Recall that, if $\Gamma_1, \dots, \Gamma_l$ are the connected components of a Coxeter graph Γ , then

$$W_\Gamma = W_{\Gamma_1} \times W_{\Gamma_2} \times \cdots \times W_{\Gamma_l}.$$

In particular, Γ is of spherical type if and only if all the components $\Gamma_1, \dots, \Gamma_l$ are of spherical type. So, we only need to classify the connected Coxeter graphs of spherical type.

- Theorem 3.8** (Coxeter [55], [56]). (1) *A Coxeter graph Γ is of spherical type if and only if the canonical bilinear form $\langle, \rangle : V \times V \rightarrow \mathbb{R}$ is positive definite.*
- (2) *The connected spherical type Coxeter graphs are the Coxeter graphs listed in Figure 3.3.*

3.3 Artin monoids

Let Γ be a Coxeter graph, let (W, S) be the Coxeter system of type Γ , and let (G, Σ) be the Artin system of type Γ . Define the *Artin monoid of type Γ* to be the monoid $G^+ = G_\Gamma^+$ presented as a monoid by the generating set $\Sigma = \{\sigma_s; s \in S\}$ and the relations

$$\begin{aligned} \text{prod}(\sigma_s, \sigma_t : m_{st}) &= \text{prod}(\sigma_t, \sigma_s : m_{st}) \quad \text{for all } s, t \in S, \\ s &\neq t \text{ and } m_{st} \neq +\infty. \end{aligned}$$

Theorem 3.9 (Paris [132]). *The natural homomorphism $G_\Gamma^+ \rightarrow G_\Gamma$ is injective.*

Recall the homomorphism $\theta : G_\Gamma \rightarrow W_\Gamma$, $\sigma_s \mapsto s$. We denote by $\theta^+ : G_\Gamma^+ \rightarrow W_\Gamma$ the restriction of θ to G_Γ^+ . We define a set-section $\kappa : W_\Gamma \rightarrow G_\Gamma^+$ of θ^+ as

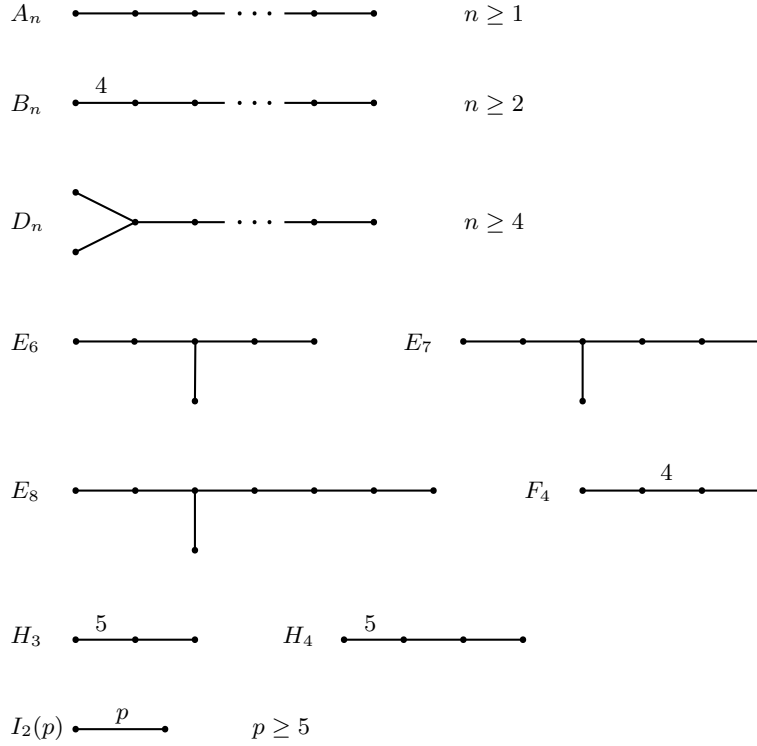


Figure 3.3. The connected spherical type Coxeter graphs.

follows. Let $w \in W$, and let $\omega = (s_1, s_2, \dots, s_l)$ be a reduced expression of w . Then

$$\kappa(w) = \sigma_{s_1} \sigma_{s_2} \cdots \sigma_{s_l}.$$

By Theorem 3.6, the definition of $\kappa(w)$ does not depend on the choice of the reduced expression of w .

Observe also that the defining relations of G_Γ^+ are homogeneous, thus G_Γ^+ has a well-defined word length $\lg : G_\Gamma^+ \rightarrow \mathbb{N}$, $\sigma_{s_1} \cdots \sigma_{s_l} \mapsto l$. This word length satisfies the following properties:

- $\lg(\alpha) = 0$ if and only if $\alpha = 1$;
- $\lg(\alpha\beta) = \lg(\alpha) + \lg(\beta)$ for all $\alpha, \beta \in G_\Gamma^+$.

We define partial orders \leq_L and \leq_R on G_Γ^+ by

- $\alpha \leq_L \beta$ if there exists $\gamma \in G_\Gamma^+$ such that $\alpha\gamma = \beta$;
- $\alpha \leq_R \beta$ if there exists $\gamma \in G_\Gamma^+$ such that $\gamma\alpha = \beta$.

The following is again a direct consequence of Theorem 3.6.

Lemma 3.10. *Let $u, v \in W$. We have $u \leq_L v$ if and only if $\kappa(u) \leq_L \kappa(v)$.*

The set $\mathcal{S} = \{\kappa(w); w \in W\}$ is called the set of *simple elements* of G_Γ^+ . If W is finite and w_0 is the greatest element of W , then $\kappa(w_0)$ is called the *Garside element* of G_Γ^+ and is denoted by $\Delta = \kappa(w_0)$.

The following theorems 3.11 and 3.12 are key results in the study of Artin monoids and groups. They are implicit in the work of Brieskorn and Saito [32], and explicit for the spherical type Artin groups in the work of Deligne [71]. Complete and detailed proofs of them can be found in [126].

Theorem 3.11. *Let $\alpha \in G_\Gamma^+$. Set*

$$E(\alpha) = \{a \in \mathcal{S}; a \leq_L \alpha\}.$$

Then $E(\alpha)$ has a greatest element. That is, there exists $a_0 \in E(\alpha)$ such that $E(\alpha) = \{a \in \mathcal{S}; a \leq_L a_0\}$.

For $\alpha \in G_\Gamma^+$ we denote by $\delta(\alpha)$ the greatest element of $E(\alpha)$.

Theorem 3.12. *Let $\alpha, \beta \in G_\Gamma^+$. Then $\delta(\alpha\beta) = \delta(\alpha\delta(\beta))$.*

Theorems 3.11 and 3.12 have the following consequences whose significance will become clear in the next section.

Theorem 3.13. (1) *Let $\alpha, \beta \in G_\Gamma^+$. There exists a unique $\gamma^\circ \in G_\Gamma^+$ such that $\gamma^\circ \leq_L \alpha$, $\gamma^\circ \leq_L \beta$, and $\gamma \leq_L \gamma^\circ$ whenever $\gamma \leq_L \alpha$ and $\gamma \leq_L \beta$.*
 (2) *Suppose that Γ is of spherical type. Let $\alpha, \beta \in G_\Gamma^+$. There exists a unique $\gamma_o \in G_\Gamma^+$ such that $\alpha \leq_L \gamma_o$, $\beta \leq_L \gamma_o$, and $\gamma_o \leq_L \gamma$ whenever $\alpha \leq_L \gamma$ and $\beta \leq_L \gamma$.*

The element γ° of Theorem 3.13 is denoted by $\gamma^\circ = \alpha \wedge_L \beta$, and the element γ_o is denoted by $\gamma_o = \alpha \vee_L \beta$ (if it exists). Note that the same result is valid if we replace \leq_L by \leq_R .

Proof. We prove (1) by induction on $\lg(\alpha) + \lg(\beta)$. By Proposition 3.7 and by Lemma 3.10, $\alpha \wedge_L \beta$ exists if $\alpha, \beta \in \mathcal{S}$.

Let $\alpha, \beta \in G_\Gamma^+$. Set $a = \delta(\alpha) \wedge_L \delta(\beta)$ (a exists by the above observation). If $a = 1$, then we must have $\gamma^\circ = \alpha \wedge_L \beta = 1$. Suppose $a \neq 1$. Let $\alpha', \beta' \in G_\Gamma^+$ such that $\alpha = a\alpha'$ and $\beta = a\beta'$. The element $\alpha' \wedge_L \beta'$ exists by induction. Then $\gamma^\circ = a \cdot (\alpha' \wedge_L \beta')$ (the proof of this equality is left to the reader).

Now, we assume that Γ is of spherical type and turn to prove (2). Let w_0 be the greatest element of W , and let $\Delta = \kappa(w_0)$ be the Garside element of G_Γ^+ . It is shown in [28] that $w_0^{-1} = w_0$ and $w_0 S w_0 = S$. This implies that $\Delta \cdot \Sigma \cdot \Delta^{-1} = \Sigma$, and, consequently, there exists a permutation $\tau : \mathcal{S} \rightarrow \mathcal{S}$ such that $\Delta\alpha = \tau(\alpha)\Delta$ for all $\alpha \in G_\Gamma^+$.

Let $\alpha \in G_\Gamma^+$. Set $\alpha = a_1 a_2 \cdots a_r$, where $a_i = \delta(a_i a_{i+1} \cdots a_r) \in \mathcal{S}$ for all $1 \leq i \leq r$. Using the above observation, it is easily shown that $\alpha \leq_L \Delta^r$.

Let $\alpha, \beta \in G_\Gamma^+$. Set $\mathcal{E} = \{\gamma \in G_\Gamma^+; \alpha \leq_L \gamma \text{ and } \beta \leq_L \gamma\}$. We have $\mathcal{E} \neq \emptyset$ since, by the above, it contains an element of the form Δ^r . Let γ_o be the smallest element of \mathcal{E} (this element exists by (1)). Then $\gamma_o = \alpha \vee_L \beta$. \square

3.4 Artin groups

We turn now to present a geometrical interpretation of the Artin groups which extends the interpretation of the braid groups in term of configuration spaces. We focus our presentation on the spherical type Artin groups, but many of the results stated in this subsection can be extended in some sense to the other Artin groups.

Let Γ be a spherical type Coxeter graph, let (W, S) be the Coxeter system of type Γ , and let (G, Σ) be the Artin system of type Γ . Recall the set $\Pi = \{e_s; s \in S\}$ of simple roots, the vector space $V = \oplus_{s \in S} \mathbb{R}e_s$, and the canonical bilinear form $\langle, \rangle : V \times V \rightarrow \mathbb{R}$, which, by Theorem 3.8, is positive definite. We assume that W is embedded in $GL(V)$ via the canonical representation.

Let \mathcal{R} be the set of reflections in W . For each $r \in \mathcal{R}$, let H_r be the hyperplane of V fixed by r . Then W acts freely on the complement of $\cup_{r \in \mathcal{R}} H_r$ (see [28]). Complexifying the action, we get an action of W on $V_{\mathbb{C}} = \mathbb{C} \otimes V$ which is free on the complement of $\cup_{r \in \mathcal{R}} \mathbb{C} \otimes H_r$. Set

$$M_\Gamma = V_{\mathbb{C}} \setminus \left(\bigcup_{r \in \mathcal{R}} \mathbb{C} \otimes H_r \right), \quad N_\Gamma = M_\Gamma / W.$$

By a theorem of Chevalley [49], Shephard, and Todd [143], $V_{\mathbb{C}}/W$ is isomorphic to \mathbb{C}^n , thus N_Γ is the complement in \mathbb{C}^n of an algebraic set, $(\cup_{r \in \mathcal{R}} \mathbb{C} \otimes H_r)/W$, called the *discriminant of type Γ* .

Theorem 3.14 (Brieskorn [30]). $\pi_1(N_\Gamma) \simeq G_\Gamma$.

Note. Infinite Coxeter groups also act as reflection groups on \mathbb{R}^n . However, to extend Theorem 3.14 to these groups we should replace V by the Tits cone $U \subset V$ (see [28]), and $V_{\mathbb{C}}$ by $(U + iV) \subset V_{\mathbb{C}}$. Then W acts freely on $(U + iV) \setminus (\cup_{r \in \mathcal{R}} \mathbb{C} \otimes H_r) = M_\Gamma$, and it was shown by Van der Lek [116] that $\pi_1(N_\Gamma) \simeq G_\Gamma$, where $N_\Gamma = M_\Gamma/W$.

An extension of Corollary 2.12 to the spherical type Artin groups is:

Theorem 3.15 (Deligne [71]). *Let Γ be a spherical type Coxeter graph. Then N_Γ and M_Γ are $K(\pi, 1)$.*

Note. It is an open problem to know whether N_Γ is $K(\pi, 1)$ if Γ is not of spherical type. The answer is yes for the so-called FC-type Artin groups and 2-dimensional Artin groups [45], and also for few affine type Artin groups (see [47], [38]).

Note. We may replace W by a finite complex reflection group acting on \mathbb{C}^n , and M_Γ by $M(W) = \mathbb{C}^n \setminus (\cup_{r \in \mathcal{R}} H_r)$, where \mathcal{R} is the set of reflections in W , and H_r denotes the hyperplane fixed by r . Here again, the group W acts freely on $M(W)$ and, by [49] and [143], $N(W) = M(W)/W$ is isomorphic to the complement in \mathbb{C}^n of an algebraic set. It was recently proved by Bessis [15] that $N(W)$ is always $K(\pi, 1)$. A classification of the finite complex reflection groups was obtained by Shephard and Todd [143], and a nice presentation of $\pi_1(N(W))$ is known for all these groups but four exceptional cases (see [33], [16]).

4 Garside groups

4.1 Garside monoids

A monoid M is called *atomic* if there exists a function $\nu : M \rightarrow \mathbb{N}$ such that

- $\nu(\alpha) = 0$ if and only if $\alpha = 1$;
- $\nu(\alpha\beta) \geq \nu(\alpha) + \nu(\beta)$ for all $\alpha, \beta \in M$.

Such a function ν is called a *norm* on M . An element $\alpha \in M$ is called an *atom* if it is indecomposable, that is, if $\alpha = \beta\gamma$, then either $\beta = 1$ or $\gamma = 1$.

The following is proved in [70].

Lemma 4.1. *Let M be an atomic monoid. A subset $S \subset M$ generates M if and only if it contains all the atoms. In particular, M is finitely generated if and only if it contains finitely many atoms.*

Let M be an atomic monoid. We define on M two partial orders \leq_L and \leq_R as follows.

- Set $\alpha \leq_L \beta$ if there exists $\gamma \in M$ such that $\alpha\gamma = \beta$.
- Set $\alpha \leq_R \beta$ if there exists $\gamma \in M$ such that $\gamma\alpha = \beta$.

The orders \leq_L and \leq_R are called the *left and right divisibility orders*, respectively.

A monoid M is called a *Garside monoid* if

- M is atomic and finitely generated;
- M is cancelative (that is, if $\alpha\beta\gamma = \alpha\beta'\gamma$, then $\beta = \beta'$, for all $\alpha, \beta, \beta', \gamma \in M$);

- (M, \leq_L) and (M, \leq_R) are lattices;
- there exists an element $\Delta \in M$, called a *Garside element*, such that the sets $L(\Delta) = \{\alpha \in M; \alpha \leq_L \Delta\}$ and $R(\Delta) = \{\alpha \in M; \alpha \leq_R \Delta\}$ are equal and generate M .

If M is a Garside monoid, then the lattice operations of (M, \leq_L) (resp. of (M, \leq_R)) are denoted by \vee_L and \wedge_L (resp. by \vee_R and \wedge_R).

Let M be a monoid. The *group of fractions* of M is defined to be the group $G(M)$ presented with the generating set M and the relations $\alpha \cdot \beta = \gamma$ if $\alpha\beta = \gamma$ in M . Such a group has the universal property that, if $\varphi : M \rightarrow H$ is a homomorphism and H is a group, then there exists a unique homomorphism $\hat{\varphi} : G(M) \rightarrow H$ such that $\varphi = \hat{\varphi} \circ \iota$, where $\iota : M \rightarrow G(M)$ is the natural homomorphism. Note that the latter homomorphism $\iota : M \rightarrow G(M)$ is not injective in general.

A *Garside group* is defined to be the group of fractions of a Garside monoid.

Note. Garside monoids and groups were introduced in [70] in a slightly restricted sense, and in [67] in the larger sense which is now generally used. This notion was extended to the notion of quasi-Garside monoids [75], [13], to study some non-spherical Artin groups. Quasi-Garside monoids have the same definition as the Garside monoids except they are not required to be finitely generated. Recently, this notion was extended to the notion of Garside categories [107], [108], [76], [14], which, in some sense, has to be considered as a geometric object more than as an algebraic one. Garside categories are a central concept in Bessis' solution to the $K(\pi, 1)$ problem for complex reflection arrangements (see [15]).

Motivating examples of Garside groups are the Artin groups of spherical type:

Theorem 4.2. *Let Γ be a spherical type Coxeter graph. Then G_Γ^+ is a Garside monoid. In particular, G_Γ is a Garside group.*

Note that Theorem 4.2 is essentially a restatement of Theorem 3.13.

Other interesting examples of Garside groups include all torus link groups (see [136]) and some generalized braid groups associated to complex reflection groups (see [15]).

Note. Two different Garside monoids can have the same group of fractions. In particular, the Artin groups of spherical type are groups of fractions of other Garside monoids, called *dual Artin monoids*, introduced by Birman, Ko, and Lee [27] for the braid groups, and by Bessis [12] for the other ones.

Note. A Garside element is not unique. For instance, if Δ is a Garside element, then Δ^k is a Garside element for all $k \geq 1$ (see [67]).

We say that a monoid M satisfies the *Öre conditions* if

- M is cancelative;
- for all $\alpha, \beta \in M$, there exist $\alpha', \beta' \in M$ such that $\alpha\alpha' = \beta\beta'$.

It is well-known that a monoid which satisfies the Öre conditions embeds in its group of fractions. On the other hand, A Garside monoid clearly satisfies the Öre conditions. Thus:

Proposition 4.3. *Let M be a Garside monoid. Then the natural homomorphism $\iota : M \rightarrow G(M)$ is injective.*

Let M be a Garside monoid and let $G = G(M)$ be the group of fractions of M . Then the partial orders \leq_L and \leq_R can be extended to G as follows.

- Set $\alpha \leq_L \beta$ if $\alpha^{-1}\beta \in M$.
- Set $\alpha \leq_R \beta$ if $\beta\alpha^{-1} \in M$.

One can easily verify that (G, \leq_L) and (G, \leq_R) are lattices. This can be used, for example, to prove the following.

Proposition 4.4. *A Garside group is torsion free.*

Proof. Let $\alpha \in G$ such that $\alpha^n = 1$ for some $n \geq 1$. Set $\beta = 1 \vee_L \alpha \vee_L \cdots \vee_L \alpha^{n-1}$. It is easily seen that \leq_L is invariant by left multiplication. This implies that $\alpha\beta = \beta$, hence $\alpha = 1$. \square

Note. Let G be a Garside group. Finite dimensional $K(G, 1)$ (that is, $K(\pi, 1)$ spaces having G as fundamental group) were described in [69] and [46]. This implies that G is torsion free, but also more.

4.2 Reversing processes and presentations

Let Σ be a finite set. Let Σ^* be the free monoid on Σ . Recall that the elements of Σ^* are the finite sequences of elements of Σ that are called *words* on Σ . If \equiv is a congruence on Σ^* and $M = (\Sigma^* / \equiv)$, then we denote by $\Sigma^* \rightarrow M$, $\omega \mapsto \bar{\omega}$ the natural epimorphism.

Define a *complement* on Σ to be a map $f : \Sigma \times \Sigma \rightarrow \Sigma^*$ such that $f(x, x) = \epsilon$ for all $x \in \Sigma$, where ϵ denotes the empty word. To a complement f we associate two monoids:

$$\begin{aligned} M_L^f &= \langle \Sigma \mid xf(x, y) = yf(y, x) \text{ for all } x, y \in \Sigma \rangle^+; \\ M_R^f &= \langle \Sigma \mid f(y, x)x = f(x, y)y \text{ for all } x, y \in \Sigma \rangle^+. \end{aligned}$$

For $u, v \in \Sigma^*$, we use the notation $u \equiv_L^f v$ (resp. $u \equiv_R^f v$) to mean that $\bar{u} = \bar{v}$ in M_L^f (resp. in M_R^f).

Example. Let Γ be a Coxeter graph, and let $M = (m_{st})_{s,t \in S}$ be the Coxeter matrix of Γ . Suppose that $m_{st} \neq +\infty$ for all $s, t \in S$, $s \neq t$. Let $\Sigma = \{\sigma_s; s \in S\}$. Let $f : \Sigma \times \Sigma \rightarrow \Sigma^*$ be the complement defined by

$$f(\sigma_s, \sigma_t) = \text{prod}(\sigma_t, \sigma_s : m_{st} - 1).$$

Then $G_\Gamma^+ = M_L^f$.

Suppose given a complement $f : \Sigma \times \Sigma \rightarrow \Sigma^*$. Let $\Sigma^{-1} = \{x^{-1}; x \in \Sigma\}$ be the set of *inverses* of elements of Σ . Let $\omega, \omega' \in (\Sigma \sqcup \Sigma^{-1})^*$. We say that ω is *f-reversible on the left in one step* to ω' if there exist $\omega_1, \omega_2 \in (\Sigma \sqcup \Sigma^{-1})^*$ and $x, y \in \Sigma$ such that

$$\omega = \omega_1 x^{-1} y \omega_2 \quad \text{and} \quad \omega' = \omega_1 \cdot f(x, y) \cdot f(y, x)^{-1} \cdot \omega_2.$$

Note that y can be equal to x in the above definition. In that case we have $\omega = \omega_1 x^{-1} x \omega_2$ and $\omega' = \omega_1 \omega_2$. Note also that $\bar{\omega} = \bar{\omega}'$ in $G(M_L^f)$ if ω is *f-reversible on the left in one step* to ω' .

Let $p \geq 0$. We say that ω is *f-reversible on the left in p steps* to ω' if there exists a sequence $\omega = \omega_0, \omega_1, \dots, \omega_p = \omega'$ in $(\Sigma \sqcup \Sigma^{-1})^*$ such that ω_{i-1} is *f-reversible on the left in one step* to ω_i for all $1 \leq i \leq p$. The property that ω is *f-reversible on the left* to ω' is denoted by $\omega \mapsto_L^f \omega'$.

We define the *f-reversibility on the right* in the same way, replacing subwords of the form yx^{-1} by their corresponding words $f(x, y)^{-1} \cdot f(y, x)$. The property that ω is *f-reversible on the right* to ω' is denoted by $\omega \mapsto_R^f \omega'$.

A word $\omega \in (\Sigma \sqcup \Sigma^{-1})^*$ is said to be *f-reduced on the left* (resp. *f-reduced on the right*) if it is of the form $\omega = vu^{-1}$ (resp. $\omega = u^{-1}v$) with $u, v \in \Sigma^*$.

It is shown in [66] that a reversing process is confluent, namely:

Proposition 4.5 (Dehornoy [66]). *Let $f : \Sigma \times \Sigma \rightarrow \Sigma^*$ be a complement, and let $\omega \in (\Sigma \sqcup \Sigma^{-1})^*$. Suppose that there exist $p \geq 0$ and a *f-reduced* word vu^{-1} on the left such that ω is *f-reversible on the left in p steps* to vu^{-1} . Then any sequence of left *f-reversing* transformations starting from ω converges to vu^{-1} in p steps.*

Let $u, v \in \Sigma^*$. Suppose there exist $u', v' \in \Sigma^*$ such that $u^{-1}v \mapsto_L^f v'(u')^{-1}$. By Proposition 4.5, the words u' and v' are unique. Moreover, it is easily checked that we also have $v^{-1}u \mapsto_L^f u'(v')^{-1}$. In this case we set

$$u' = C_L^f(v, u) \quad \text{and} \quad v' = C_L^f(u, v).$$

Similarly, if there exist $u', v' \in \Sigma^*$ such that $vu^{-1} \mapsto_R^f (u')^{-1}(v')$, then $uv^{-1} \mapsto_R^f (v')^{-1}(u')$, u' and v' are unique, and we set

$$u' = C_R^f(u, v) \quad \text{and} \quad v' = C_R^f(v, u).$$

Lemma 4.6 (Dehornoy [66]). *Let $f : \Sigma \times \Sigma \rightarrow \Sigma^*$ be a complement. Let $u, v \in \Sigma^*$. Suppose that $C_L^f(u, v)$ and $C_L^f(v, u)$ exist. Then*

$$u \cdot C_L^f(u, v) \equiv_L^f v \cdot C_L^f(v, u).$$

A complement $f : \Sigma \times \Sigma \rightarrow \Sigma^*$ is said to be *coherent on the left* if, for all $x, y, z \in \Sigma$, $C_L^f(f(x, y), f(x, z))$ and $C_L^f(f(y, x), f(y, z))$ exist and are \equiv_L^f -equivalent. Similarly, we say that f is *coherent on the right* if, for all $x, y, z \in \Sigma$, $C_R^f(f(z, x), f(y, x))$ and $C_R^f(f(z, y), f(x, y))$ exist and are \equiv_R^f -equivalent.

Theorem 4.7 (Dehornoy, Paris [70], [67]). *Let M be a finitely generated monoid, and let Σ be a finite generating set of M . Then M is a Garside monoid if and only if it satisfies the following three conditions.*

- M is atomic.
- There exist a complement $f : \Sigma \times \Sigma \rightarrow \Sigma^*$ coherent on the left and a complement $g : \Sigma \times \Sigma \rightarrow \Sigma^*$ coherent on the right such that $M = M_L^f = M_R^g$.
- There exists an element $\Delta \in M$ such that the sets $L(\Delta) = \{\alpha \in M; \alpha \leq_L \Delta\}$ and $R(\Delta) = \{\alpha \in M; \alpha \leq_R \Delta\}$ are equal and generate M .

We refer to [70] and [67] for more “algorithmic” conditions to detect a Garside monoid in terms of complements and presentations, and turn to explain some applications of the reversing processes.

Let M be a Garside monoid, and let $G = G(M)$ be its group of fractions. Let $f : \Sigma \times \Sigma \rightarrow \Sigma^*$ and $g : \Sigma \times \Sigma \rightarrow \Sigma^*$ be complements such that $M = M_L^f = M_R^g$.

First, the complements f and g lead to algorithms:

Proposition 4.8 (Dehornoy, Paris [70], [67]). (1) *The complement f is coherent on the left, and the complement g is coherent on the right.*
 (2) *Let $\omega \in (\Sigma \sqcup \Sigma^{-1})^*$. There exist a (unique) f -reduced word vu^{-1} on the left, and a (unique) g -reduced word $(u')^{-1}(v')$ on the right, such that $\omega \mapsto_L^f vu^{-1}$ and $\omega \mapsto_R^g (u')^{-1}(v')$.*

They can be used to solve the word problem:

Proposition 4.9 (Dehornoy, Paris [70], [67]). *Let $\omega \in (\Sigma \sqcup \Sigma^{-1})^*$. Let $u, v \in \Sigma$ such that $\omega \mapsto_L^f vu^{-1}$ (see Proposition 4.8). Then $\bar{\omega} = 1$ in $G = G(M)$ if and only if $u^{-1}v \mapsto_L^f \epsilon$, where ϵ denotes the empty word.*

They can be also used to compute the lattice operations of (M, \leq_L) and (M, \leq_R) .

Proposition 4.10 (Dehornoy, Paris [70], [67]). *Let $u, v \in \Sigma^*$. Set $u' = C_L^f(u, v)$ and $v' = C_L^f(v, u)$. Then $\bar{u} \vee_L \bar{v}$ is represented by*

$$uu' \equiv_L^f vv',$$

and $\bar{u} \wedge_L \bar{v}$ is represented by

$$C_R^g(u, C_R^g(v', u')) \equiv_L^f C_R^g(v, C_R^g(u', v')).$$

4.3 Normal forms and automatic structures

Let M be a Garside monoid, let $G = G(M)$ be the group of fractions of M , and let Δ be a fixed Garside element of M . Define the set of *simple elements* to be

$$\mathcal{S} = \{a \in M ; a \leq_L \Delta\} = \{a \in M ; a \leq_R \Delta\}.$$

By definition, \mathcal{S} is finite and generates M .

Let $\alpha \in M$. Then α can be uniquely written in the form

$$\alpha = a_1 a_2 \cdots a_l,$$

where $a_1, a_2, \dots, a_l \in \mathcal{S}$, and

$$a_i = \Delta \wedge_L (a_i a_{i+1} \cdots a_l) \quad \text{for all } 1 \leq i \leq l.$$

Such an expression of α is called the *normal form* of α .

Let $\alpha \in G$. Then α can be written in the form $\alpha = \beta^{-1} \gamma$, where $\beta, \gamma \in M$ (see Proposition 4.8, for instance). Obviously, we can also assume that $\beta \wedge_L \gamma = 1$. In that case β and γ are unique. Let $\beta = b_1 b_2 \cdots b_p$ be the normal form of β and let $\gamma = c_1 c_2 \cdots c_q$ be the normal form of γ . Then the expression

$$\alpha = b_p^{-1} \cdots b_2^{-1} b_1^{-1} c_1 c_2 \cdots c_q$$

is called the *normal form* of α .

There is another notion of normal forms for the elements of G , called Δ -normal forms, that are used, in particular, in several solutions to the conjugacy problem for G . They are defined as follows.

It is easily seen that there exists a permutation $\tau : \mathcal{S} \rightarrow \mathcal{S}$ such that $\Delta a \Delta^{-1} = \tau(a)$ for all $a \in \mathcal{S}$. Moreover, for all $a \in \mathcal{S}$, there exists $a^* \in \mathcal{S}$ such that $a^* a = \Delta$ (i.e. $a^{-1} = \Delta^{-1} a^*$). These two observations show that every $\alpha \in G$ can be written in the form $\alpha = \Delta^p \beta$, where $p \in \mathbb{Z}$ and $\beta \in M$. One can choose p to be maximal, and, in that case, β is unique. Let $b_1 b_2 \cdots b_r$ be the normal form of β . Then the expression

$$\alpha = \Delta^p b_1 b_2 \cdots b_r$$

is called the Δ -normal form of α .

A *finite state automaton* is a quintuple $\mathcal{A} = (Q, \mathcal{S}, T, A, q_0)$, where

- Q is a finite set, called the set of *states*;
- \mathcal{S} is a finite set, called the *alphabet*;
- T is a map $T : Q \times \mathcal{S} \rightarrow Q$, called the *transition function*;
- A is a subset of Q , called the set of *accepted states*;
- q_0 is an element of Q , called the *initial state*.

The *iterated transition function* is the map $T^* : Q \times \mathcal{S}^* \rightarrow Q$ defined by induction on the length of the second component as follows.

$$\begin{aligned} T^*(q, \epsilon) &= q, \\ T^*(q, x_1 x_2 \cdots x_l) &= T(T^*(q, x_1 \cdots x_{l-1}), x_l). \end{aligned}$$

The set

$$\mathcal{L}_A = \{\omega \in \mathcal{S}^* ; T^*(q_0, \omega) \in A\}$$

is called the *language recognized by \mathcal{A}* . A *regular language* is a language recognized by a finite state automaton.

Let G be a group generated by a finite set \mathcal{S} . Define the *word length* of an element $\alpha \in G$, denoted by $\text{lg}_{\mathcal{S}}(\alpha)$, to be the shortest length of a word in $(\mathcal{S} \sqcup \mathcal{S}^{-1})^*$ which represents α . The *distance* between two element $\alpha, \beta \in G$, denoted by $d_{\mathcal{S}}(\alpha, \beta)$, is the length of $\alpha^{-1}\beta$.

Let $\mathcal{L} \subset (\mathcal{S} \sqcup \mathcal{S}^{-1})^*$ be a language. We say that \mathcal{L} *represents G* if every element of G is represented by an element of \mathcal{L} . We say, furthermore, that \mathcal{L} has the *uniqueness property* if every element of G is represented by a unique element of \mathcal{L} . We say that \mathcal{L} is *symmetric* if $\mathcal{L}^{-1} = \mathcal{L}$, where $\mathcal{L}^{-1} = \{\omega^{-1}; \omega \in \mathcal{L}\}$. We say that \mathcal{L} is *geodesic* if $\text{lg}(\omega) = \text{lg}_{\mathcal{S}}(\bar{\omega})$ for all $\omega \in \mathcal{L}$. Let $\omega = x_1^{\epsilon_1} \cdots x_l^{\epsilon_l} \in (\mathcal{S} \sqcup \mathcal{S}^{-1})^*$. For $t \in \mathbb{N}$ we set

$$\bar{\omega}(t) = \begin{cases} 1 & \text{if } t = 0 \\ \overline{x_1^{\epsilon_1} \cdots x_t^{\epsilon_t}} & \text{if } 1 \leq t \leq l \\ \bar{\omega} & \text{if } t \geq l \end{cases}$$

Let c be a positive integer. We say that \mathcal{L} has the *c-fellow traveler property* if

$$d_{\mathcal{S}}(\bar{u}(t), \bar{v}(t)) \leq c \cdot d_{\mathcal{S}}(\bar{u}, \bar{v})$$

for all $u, v \in \mathcal{L}$ and all $t \in \mathbb{N}$.

A group G is said to be *automatic* if there exist a finite generating set $\mathcal{S} \subset G$, a regular language $\mathcal{L} \subset (\mathcal{S} \sqcup \mathcal{S}^{-1})^*$, and a constant $c > 0$, such that \mathcal{L} represents G and has the c -fellow traveler property. If, in addition, \mathcal{L}^{-1} has also the c -fellow traveler property, then G is said to be *biautomatic*. We say that G is *fully biautomatic* if \mathcal{L} is symmetric, and that G is *geodesically automatic* if \mathcal{L} is geodesic.

Biautomatic groups have many attractive properties. For instance, they have soluble word and conjugacy problems, and they have quadratic isoperimetric inequalities. We refer to [79] for a general exposition on the subject.

Theorem 4.11 (Charney [43], Dehornoy, Paris [70]). *Let M be a Garside monoid, and let $G = G(M)$ be the group of fractions of M . Let $\mathcal{L} \subset (\mathcal{S} \sqcup \mathcal{S}^{-1})^*$ be the language of normal forms. Then \mathcal{L} is regular, represents G , has the uniqueness property, has the 5-fellow traveler property, is symmetric, and is geodesic.*

Corollary 4.12. *Garside groups are fully geodesically biautomatic.*

Note. The language of Δ -normal forms is also regular and satisfies some fellow traveler property, and the language of inverses of Δ -normal forms satisfies the same fellow traveler property. So, Δ -normal forms determine another biautomatic structure on G . This was proved by Thurston [79] for the braid groups and by Charney [42] for all the spherical type Artin groups, but the same proof works in general for all Garside groups.

4.4 Conjugacy problem

Let G be a group and let \mathcal{S} be a finite generating set of G . A *solution to the conjugacy problem* for G is an algorithm which, for given $u, v \in (\mathcal{S} \sqcup \mathcal{S}^{-1})^*$, decides whether \bar{u} and \bar{v} are conjugate or not.

The first solution to the conjugacy problem for the braid groups was obtained by Garside [85]. Garside's algorithm was improved by El-Rifai and Morton [78], and this improvement was extended to the Garside groups by Picantin [135]. Picantin's algorithm was improved by Franco and González-Meneses [81], then by Gebhardt [86], and now by Gebhardt and González-Meneses [87]. The algorithm that we present here is not the optimal one, but is probably the simplest one. It is based on the algorithm of [87].

Note. In addition to the above mentioned papers, there are several recent papers where the algorithms are analyzed, in particular to obtain the best possible complexity (see [22], [23], [24], [89], [115], [113], [114]). These analysis often lead to new and unexpected results on the braid groups and, more generally, on the Garside groups.

Let M be a Garside monoid, let $G = G(M)$ be its group of fractions, let Δ be a fixed Garside element, and let $\mathcal{S} = \{a \in M; a \leq_L \Delta\}$ be the set of simple elements. Recall that, for every $a \in \mathcal{S}$, there exists a unique $a^* \in \mathcal{S}$ such that $aa^* = \Delta$. Recall also that there exists a permutation $\tau : \mathcal{S} \rightarrow \mathcal{S}$ such that $\Delta a \Delta^{-1} = \tau(a)$ for all $a \in \mathcal{S}$.

Let $\alpha \in G$. Let $\alpha = \Delta^p a_1 a_2 \cdots a_r$ be the Δ -normal form of α . The number p is called the *infimum* of α and is denoted by $\inf(\alpha)$, $p + r$ is called the *supremum* and is denoted by $\sup(\alpha)$, and r is called the *canonical length* and is denoted by $\|\alpha\|$. The above terminology comes from the fact that p is the

greatest number n such that $\Delta^n \leq_L \alpha$, and $p + r$ is the smallest number n such that $\alpha \leq_L \Delta^n$. The (simple) element $\tau^p(a_1)$ is called the *initial factor* of α and is denoted by $i(\alpha)$, and a_r is called the *terminal factor* and is denoted by $t(\alpha)$. It is easily checked that $i(\alpha^{-1}) = t(\alpha)^*$. Let

$$\pi(\alpha) = i(\alpha) \wedge_L t(\alpha)^* = i(\alpha) \wedge_L i(\alpha^{-1}).$$

Define the *sliding* of α to be

$$S(\alpha) = \pi(\alpha)^{-1} \cdot \alpha \cdot \pi(\alpha).$$

Observe that $\|S(\alpha)\| \leq \|\alpha\|$.

For $\alpha, \beta \in G$, we use the notation $\alpha \sim \beta$ to mean that α is conjugate to β . Let $\alpha \in G$. Define the *sliding circuits* of α to be

$$SC(\alpha) = \{\beta \in G ; \beta \sim \alpha \text{ and } S^m(\beta) = \beta \text{ for some } m \geq 1\}.$$

It is shown in [87] that the elements of $SC(\alpha)$ have minimal canonical length in the conjugacy class of α , but not all the elements of the conjugacy class of minimal canonical length belong to $SC(\alpha)$.

Clearly, if $\alpha \sim \beta$, then $SC(\alpha) = SC(\beta)$, and if $\alpha \not\sim \beta$, then $SC(\alpha) \cap SC(\beta) = \emptyset$. So, our solution to the conjugacy problem for G follows the following stages.

Input. Two elements $\alpha, \beta \in G$.

Stage 1. Calculate an element $\alpha_0 \in SC(\alpha)$ and an element $\beta_0 \in SC(\beta)$.

Stage 2. Calculate the whole set $SC(\alpha) = SC(\alpha_0)$ from α_0 .

Output. YES if $\beta_0 \in SC(\alpha)$, and NO otherwise.

In order to find an element of $SC(\alpha)$ we use the following which is easy to prove.

Lemma 4.13. *Let $\alpha \in G$. There exists $m, k \geq 1$ such that $S^{m+k}(\alpha) = S^k(\alpha)$. In particular, $S^k(\alpha) \in SC(\alpha)$.*

The key result for Stage 2 is the following.

Theorem 4.14 (Gebhardt, González-Meneses [87]). *Let $\alpha, \beta \in G$ and let $\gamma_1, \gamma_2 \in M$. If $\beta, \gamma_1^{-1}\beta\gamma_1$, and $\gamma_2^{-1}\beta\gamma_2$ are elements of $SC(\alpha)$, then $(\gamma_1 \wedge_L \gamma_2)^{-1}\beta(\gamma_1 \wedge_L \gamma_2)$ is also an element of $SC(\alpha)$.*

Corollary 4.15. *Let $\alpha, \beta, \gamma \in G$ such that β and $\gamma^{-1}\beta\gamma$ are elements of $SC(\alpha)$. Let $\gamma = \Delta^p c_1 c_2 \cdots c_r$ be the Δ -normal form of γ . Set $\beta_0 = \Delta^{-p}\beta\Delta^p$, and $\beta_i = c_i^{-1}\beta_{i-1}c_i$ for $1 \leq i \leq r$. Then $\beta_i \in SC(\alpha)$ for all $0 \leq i \leq r$.*

Proof. We prove that $\beta_i \in SC(\alpha)$ by induction on i . It is easily seen that, if $\beta \in SC(\alpha)$, then $\Delta^{-1}\beta\Delta \in SC(\alpha)$. In particular, we have $\beta_0 = \Delta^{-p}\beta\Delta^p \in SC(\alpha)$.

Let $i > 0$. By induction, $\beta_{i-1} \in SC(\alpha)$. By the above observation, we have $\Delta^{-1}\beta_{i-1}\Delta \in SC(\alpha)$. On the other hand, we have $\gamma^{-1}\beta\gamma = (c_i c_{i+1} \cdots c_r)^{-1} \beta_{i-1} (c_i c_{i+1} \cdots c_r) \in SC(\alpha)$. By definition of a normal form, we have $\Delta \wedge_L (c_i c_{i+1} \cdots c_r) = c_i$. We conclude by Theorem 4.14 that $\beta_i = c_i^{-1} \beta_{i-1} c_i \in SC(\alpha)$. \square

From Corollary 4.15 we obtain the following which, together with Lemma 4.13, provides an algorithm to compute $SC(\alpha)$.

Corollary 4.16. *Let $\alpha \in G$. Let Ω_α be the graph defined by the following data.*

- *The set of vertices of Ω_α is $SC(\alpha)$.*
- *Two vertices $\beta, \beta' \in SC(\alpha)$ are joined by an edge if there exists $a \in S$ such that $\beta' = a^{-1}\beta a$.*

Then Ω_α is connected.

5 Cohomology and Salvetti complex

5.1 Cohomology

Let Γ be a Coxeter graph, let (W_Γ, S) be the Coxeter system of type Γ , and let (G_Γ, Σ) be the Artin system of type Γ . Let Γ_{ab} be the graph defined by the following data.

- S is the set of vertices of Γ ;
- two vertices $s, t \in S$ are joined by an edge if $m_{st} \neq +\infty$ and m_{st} is odd.

The following is easy to prove from the presentation of G_Γ .

Proposition 5.1. *Let d be the number of connected components of Γ_{ab} . Then the abelianization of G_Γ is a free abelian group of rank d . In particular, $H^1(G_\Gamma, \mathbb{Z}) \simeq \mathbb{Z}^d$.*

Now, assume that Γ is of spherical type, and recall the space N_Γ defined in Subsection 3.4. Except Proposition 5.1, all known results on the cohomology of G_Γ use the fact that $\pi_1(N_\Gamma) = G_\Gamma$ (see Theorem 3.14), and N_Γ is a $K(\pi, 1)$ space (see Theorem 3.15). Recall that these two results imply that $H^*(G_\Gamma, A) = H^*(N_\Gamma, A)$ for any G_Γ -module A .

In [4] Arnol'd established the following properties on the cohomology of the braid groups.

Theorem 5.2 (Arnol'd [4]). *Let $n \geq 2$.*

- (1) $H^0(\mathcal{B}_n, \mathbb{Z}) = H^1(\mathcal{B}_n, \mathbb{Z}) = \mathbb{Z}$, $H^q(\mathcal{B}_n, \mathbb{Z})$ is finite for all $q \geq 2$, and $H^q(\mathcal{B}_n, \mathbb{Z}) = 0$ for all $q \geq n$.
- (2) If n is even, then $H^q(\mathcal{B}_n, \mathbb{Z}) = H^q(\mathcal{B}_{n+1}, \mathbb{Z})$ for all $q \geq 0$.
- (3) $H^q(\mathcal{B}_n, \mathbb{Z}) = H^q(\mathcal{B}_{2q-2}, \mathbb{Z})$ for all $q \leq \frac{1}{2}n + 1$.

The study of the cohomology of the braid groups was continued by Fuks [84] who calculated the cohomology of \mathcal{B}_n with coefficients in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Let $\mathcal{B}_\infty = \varinjlim \mathcal{B}_n$, where the limit is taken relative to the natural embeddings $\mathcal{B}_n \hookrightarrow \mathcal{B}_{n+1}$, $n \geq 2$.

Theorem 5.3 (Fuks [84]). (1) $H^*(\mathcal{B}_\infty, \mathbb{F}_2)$ is the exterior \mathbb{F}_2 -algebra generated by $\{a_{m,k}; m \geq 1 \text{ and } k \geq 0\}$ where $\deg a_{m,k} = 2^k(2^m - 1)$.

- (2) The natural embedding $\mathcal{B}_n \hookrightarrow \mathcal{B}_\infty$ induces a surjective homomorphism $H^*(\mathcal{B}_\infty, \mathbb{F}_2) \rightarrow H^*(\mathcal{B}_n, \mathbb{F}_2)$ whose kernel is generated by the monomials

$$a_{m_1, k_1} a_{m_2, k_2} \cdots a_{m_t, k_t}$$

such that

$$2^{m_1 + \cdots + m_t + k_1 + \cdots + k_t} > n.$$

Later, the cohomology with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (where p is an odd prime number) and the cohomology with coefficients in \mathbb{Z} were calculate by Cohen [53], Segal [142], and Vainšteĭn [148].

Theorem 5.4 (Cohen [53], Segal [142], Vainšteĭn [148]). (1) $H^*(\mathcal{B}_\infty, \mathbb{F}_p)$ is the tensor product of a polynomial algebra generated by $\{x_i; i \geq 0\}$, where $\deg x_i = 2p^{i+1} - 2$, and an exterior algebra generated by $\{y_j; j \geq 0\}$, where $\deg y_j = 2p^j - 1$.

- (2) The natural embedding $\mathcal{B}_n \hookrightarrow \mathcal{B}_\infty$ induces a surjective homomorphism $H^*(\mathcal{B}_\infty, \mathbb{F}_p) \rightarrow H^*(\mathcal{B}_n, \mathbb{F}_p)$, whose kernel is generated by the monomials

$$x_{i_1} x_{i_2} \cdots x_{i_s} y_{j_1} y_{j_2} \cdots y_{j_t}$$

such that

$$2(p^{i_1+1} + \cdots + p^{i_s+1} + p^{j_1} + \cdots + p^{j_t}) > n.$$

Let $\beta_2 : H^*(\mathcal{B}_n, \mathbb{F}_2) \rightarrow H^*(\mathcal{B}_n, \mathbb{F}_2)$ be the homomorphism defined by

$$\beta_2(a_{m,k}) = a_{m+1,0} a_{m,1} \cdots a_{m,k-1}.$$

For an odd prime number p , let $\beta_p : H^*(\mathcal{B}_n, \mathbb{F}_p) \rightarrow H^*(\mathcal{B}_n, \mathbb{F}_p)$ be the homomorphism defined by

$$\beta_p(x_i) = y_{i+1}, \quad \beta_p(y_j) = 0.$$

Theorem 5.5 (Cohen [53], Vainšteĭn [148]). *Let $q \geq 2$. Then*

$$H^q(\mathcal{B}_n, \mathbb{Z}) = \bigoplus_p \beta_p(H^{q-1}(\mathcal{B}_n, \mathbb{F}_p)),$$

where the sum is over all primes p .

The integral cohomology of the Artin groups of type B and D were calculated by Goryunov [91] in terms of the cohomology groups of the braid groups.

Theorem 5.6 (Goryunov [91]). (1) *Let $n \geq 2$, and let $q \geq 2$. Then*

$$H^q(G_{B_n}, \mathbb{Z}) = \bigoplus_{i=0}^n H^{q-i}(\mathcal{B}_{n-i}, \mathbb{Z}).$$

(2) *Let $n \geq 4$, and let $q \geq 2$. Then*

$$H^q(G_{D_n}, \mathbb{Z}) = H^q(\mathcal{B}_n, \mathbb{Z}) \oplus \left(\bigoplus_{i=0}^{+\infty} \text{Ker } \gamma_{n-2i}^{q-2i} \right) \oplus \left(\bigoplus_{j=0}^{+\infty} H^{q-2j-3}(\mathcal{B}_{n-3j-3}, \mathbb{F}_2) \right),$$

where, for $k \geq 2$ and $j \geq 0$, $\gamma_k^j : H^j(\mathcal{B}_k, \mathbb{Z}) \rightarrow H^j(\mathcal{B}_{k-1}, \mathbb{Z})$ denotes the homomorphism induced by the inclusion $\mathcal{B}_{k-1} \hookrightarrow \mathcal{B}_k$.

Finally, the integral cohomology of the remainder irreducible Artin groups of spherical type were calculate by Salvetti in [141].

Theorem 5.7 (Salvetti [141]). *The integral cohomology of the Artin groups of type $I_2(p)$ ($p \geq 5$), E_6 , E_7 , E_8 , F_4 , H_3 , and H_4 is given in Table 5.1.*

Note. It is a direct consequence of [139] that N_Γ has the same homotopy type as a CW-complex of dimension n , where $n = |S|$. This implies that the cohomological dimension of G_Γ is $\leq n$, and, therefore, that $H^q(G_\Gamma, \mathbb{Z}) = 0$ for all $q > n$.

Note. Recall the space M_Γ of Subsection 3.4. The cohomology $H^*(M_\Gamma, \mathbb{Z})$ was calculate by Brieskorn in [31]. In particular, $H^*(M_\Gamma, \mathbb{Z})$ is torsion free and $H^n(M_\Gamma, \mathbb{Z}) \neq 0$. Let CG_Γ be the kernel of the canonical epimorphism $\theta : G_\Gamma \rightarrow W_\Gamma$. By [71] we have $H^n(M_\Gamma, \mathbb{Z}) = H^n(CG_\Gamma, \mathbb{Z})$, thus, by the above, $\text{cd}(G_\Gamma) = \text{cd}(CG_\Gamma) \geq n$, where $\text{cd}(G_\Gamma)$ denotes the cohomological dimension of G_Γ . We already know that $\text{cd}(G_\Gamma) \leq n$, thus $\text{cd}(G_\Gamma) = n$.

	H^0	H^1	H^2	H^3	H^4
$I_2(2q)$	\mathbb{Z}	\mathbb{Z}^2	\mathbb{Z}	0	0
$I_2(2q+1)$	\mathbb{Z}	\mathbb{Z}	0	0	0
H_3	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	0
H_4	\mathbb{Z}	\mathbb{Z}	0	$\mathbb{Z} \times \mathbb{Z}_2$	\mathbb{Z}
F_4	\mathbb{Z}	\mathbb{Z}^2	\mathbb{Z}^2	\mathbb{Z}^2	\mathbb{Z}
E_6	\mathbb{Z}	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2
E_7	\mathbb{Z}	\mathbb{Z}	0	\mathbb{Z}_2	$\mathbb{Z}_2 \times \mathbb{Z}_2$
E_8	\mathbb{Z}	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2

Table 5.1.(a). Cohomology of the spherical type Artin groups.

	H^5	H^6	H^7	H^8
E_6	\mathbb{Z}_6	\mathbb{Z}_3	0	0
E_7	$\mathbb{Z}_6 \times \mathbb{Z}_6$	$\mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}$	\mathbb{Z}	0
E_8	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}$	\mathbb{Z}

Table 5.1.(b). Cohomology of the spherical type Artin groups.

Note. The ring structure of $H^*(G_\Gamma, \mathbb{Z})$, where Γ is a Coxeter graph in the list of Theorem 5.7, was calculated in [111]. Some cohomologies with twisted coefficients were also considered. An interesting case is the cohomology over the module of Laurent polynomials $\mathbb{Q}[q^{\pm 1}]$ (resp. $\mathbb{Z}[q^{\pm 1}]$), because it determines the rational (resp. integral) cohomology of the Milnor fiber of the discriminant of type Γ (see [36]). For the case $\Gamma = A_n$ (i.e. G_Γ is the braid group \mathcal{B}_{n+1}), the $\mathbb{Q}[q^{\pm 1}]$ -cohomology was calculated by several people in several ways (see [82], [122], [52], [61]), and the $\mathbb{Z}[q^{\pm 1}]$ -cohomology was calculated by Callegaro in [37]. The $\mathbb{Q}[q^{\pm 1}]$ -cohomology for the other spherical type Artin groups was calculated in [62]. The $\mathbb{Z}[q^{\pm 1}]$ -cohomology for the exceptional cases was calculated in [39], and the top $\mathbb{Z}[q^{\pm 1}]$ -cohomology for all cases was calculated in [64].

Note. The cohomology of the non-spherical Artin groups is badly understood. Some calculations for the type \tilde{A}_n were done in [38].

We refer to [149] for a more detailed exposition on the cohomology of the braid groups and the Artin groups of spherical type, and turn to present the Salvetti complex (of a real hyperplane arrangement). This is the main tool in Salvetti's calculations of the cohomology of Artin groups (see [141]), but it can be used for other purposes. For instance, it can be also used to prove Theorems 3.14 and 3.15 (see [140] and [130]), and to produce a free resolution of \mathbb{Z} by $\mathbb{Z}[G_\Gamma]$ -modules (see Theorem 5.15).

5.2 Salvetti complex

Define a (*real*) *hyperplane arrangement* to be a finite family \mathcal{A} of linear hyperplanes of \mathbb{R}^n . For every $H \in \mathcal{A}$ we denote by $H_{\mathbb{C}}$ the hyperplane of \mathbb{C}^n having the same equation as H (i.e. $H_{\mathbb{C}} = \mathbb{C} \otimes H$), and we set

$$M(\mathcal{A}) = \mathbb{C}^n \setminus \left(\bigcup_{H \in \mathcal{A}} H_{\mathbb{C}} \right).$$

Note that $M(\mathcal{A})$ is an open connected subvariety of \mathbb{C}^n .

The arrangement \mathcal{A} subdivides \mathbb{R}^n into *facets*. We denote by $\mathcal{F}(\mathcal{A})$ the set of all facets. The *support* of a facet $F \in \mathcal{F}(\mathcal{A})$ is the linear subspace $\langle F \rangle$ spanned by F . We denote by \bar{F} the closure of a facet F . We order $\mathcal{F}(\mathcal{A})$ by $F \leq G$ if $F \subset \bar{G}$. The set $\mathcal{F}(\mathcal{A})$ has a unique minimal element: $\cap_{H \in \mathcal{A}} H$. The maximal elements of $\mathcal{F}(\mathcal{A})$ are the facets of codimension 0, and they are called *chambers*. The set of all chambers is denoted by $\mathcal{C}(\mathcal{A})$.

Set

$$\mathcal{X} = \{(F, C) \in \mathcal{F}(\mathcal{A}) \times \mathcal{C}(\mathcal{A}) ; F \leq C\}.$$

We partially order \mathcal{X} as follows. For $F \in \mathcal{F}(\mathcal{A})$ we set $\mathcal{A}_F = \{H \in \mathcal{A}; H \supset F\}$. For $F \in \mathcal{F}(\mathcal{A})$ and $C \in \mathcal{C}(\mathcal{A})$ we denote by C_F the chamber of \mathcal{A}_F which contains C . We set

$$(F_1, C_1) \leq (F_2, C_2) \quad \text{if} \quad F_1 \leq F_2 \quad \text{and} \quad (C_1)_{F_2} = (C_2)_{F_2}.$$

(See Figure 5.1.)

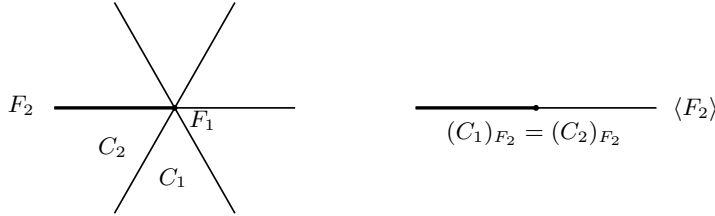


Figure 5.1. Order in \mathcal{X} .

Define the *Salvetti complex* $\text{Sal}(\mathcal{A})$ of \mathcal{A} to be the (geometric realization of the) flag complex of (\mathcal{X}, \leq) . That is, to every chain $X_0 < X_1 < \dots < X_d$ in \mathcal{X} corresponds a simplex $\Delta(X_0, X_1, \dots, X_d)$ of $\text{Sal}(\mathcal{A})$, and every simplex of $\text{Sal}(\mathcal{A})$ is of this form.

Theorem 5.8 (Salvetti [139]). *The simplicial complex $\text{Sal}(\mathcal{A})$ is homotopy equivalent to $M(\mathcal{A})$.*

We turn now to describe a cellular decomposition of $\text{Sal}(\mathcal{A})$ which is the version which is usually used in the literature.

Without loss of generality, we can and do assume that \mathcal{A} is *essential*, that is, $\cap_{H \in \mathcal{A}} H = \{0\}$. Consider the unit sphere $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| = 1\}$. The arrangement \mathcal{A} determines a cellular decomposition of \mathbb{S}^{n-1} : to each facet $F \in \mathcal{F}(\mathcal{A}) \setminus \{0\}$ corresponds the open cell $F \cap \mathbb{S}^{n-1}$, and each cell is of this form. This cellular decomposition is regular in the sense that the closure of a cell is a closed disk. Hence, one can consider the barycentric subdivision. For each facet $F \in \mathcal{F}(\mathcal{A}) \setminus \{0\}$ we fix a point $\mathbf{x}(F) \in F \cap \mathbb{S}^{n-1}$. To each chain $\{0\} \neq F_0 < F_1 < \dots < F_d$ in $\mathcal{F}(\mathcal{A}) \setminus \{0\}$ corresponds a simplex $\Delta(F_0, F_1, \dots, F_d)$ whose vertices are $\mathbf{x}(F_0), \mathbf{x}(F_1), \dots, \mathbf{x}(F_d)$, and every simplex of \mathbb{S}^{n-1} is of this form. So, the simplicial decomposition of \mathbb{S}^{n-1} is the flag complex of $(\mathcal{F}(\mathcal{A}) \setminus \{0\}, \leq)$.

We extend the above simplicial decomposition of \mathbb{S}^{n-1} to a simplicial decomposition of the n -disk $\mathbb{B}^n = \{\mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| \leq 1\}$, adding a single vertex $\mathbf{x}(0) = 0$. That is, we view \mathbb{B}^n as the cone of \mathbb{S}^{n-1} . Now, to any chain $F_0 < F_1 < \dots < F_d$ in $\mathcal{F}(\mathcal{A})$ corresponds a simplex $\Delta(F_0, F_1, \dots, F_d)$ of \mathbb{B}^n (here we may have $F_0 = 0$), and every simplex of \mathbb{B}^n is of this form. Note that this simplicial decomposition of \mathbb{B}^n is the flag complex of $(\mathcal{F}(\mathcal{A}), \leq)$.

Let $F_b \in \mathcal{F}(\mathcal{A})$ be a facet. It can be easily checked that the union of the simplices of the form $\Delta(F_0, F_1, \dots, F_d)$ with $F_b = F_0 < F_1 < \dots < F_d$ is a closed disk whose dimension is equal to $\text{codim } F_b$. Its interior is denoted by $U(F_b)$. So, the set $\{U(F); F \in \mathcal{F}(\mathcal{A})\}$ forms a cellular decomposition of \mathbb{B}^n called the *dual decomposition*.

Example. Let \mathcal{A} be a collection of 3 lines in \mathbb{R}^2 (see Figure 5.2). The poset $\mathcal{F}(\mathcal{A})$ contains 6 chambers, 6 facets of dimension 1 (half-lines), and 0. The dual decomposition of $\mathbb{B}^2 = \mathbb{D}$ has 6 vertices, 6 edges, and one 2-cell.

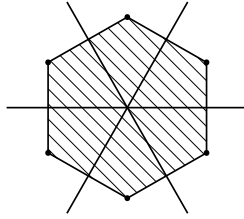


Figure 5.2. A dual decomposition.

Let $X_b = (F_b, C_b) \in \mathcal{X}$. We denote by $\bar{U}(X_b)$ the union of the simplices $\Delta(X_0, X_1, \dots, X_d)$ of $\text{Sal}(\mathcal{A})$ such that $X_b = X_0 < X_1 < \dots < X_d$. One can show (with some effort) that, for every $F \geq F_b$, there exists a unique chamber

$C \in \mathcal{C}(\mathcal{A})$ such that $F \leq C$ and $(F_b, C_b) \leq (F, C)$. This implies that $\bar{U}(X_b)$ is homeomorphic to $\bar{U}(F_b)$ via the map $(F, C) \mapsto \mathbf{x}(F)$, $F_b \leq F$. Hence, $\bar{U}(X_b)$ is a closed disk whose dimension is equal to $\text{codim } F_b$. We denote by $U(X_b)$ the interior of $\bar{U}(X_b)$. So, $\{U(X); X \in \mathcal{X}\}$ forms a (regular) cell decomposition of $\text{Sal}(\mathcal{A})$.

0-skeleton. For $C \in \mathcal{C}(\mathcal{A})$, we set $\omega(C) = U(C, C) = \bar{U}(C, C)$. Then the 0-skeleton of $\text{Sal}(\mathcal{A})$ is

$$\text{Sal}_0(\mathcal{A}) = \{\omega(C) ; C \in \mathcal{C}(\mathcal{A})\}.$$

1-skeleton. Let $F \in \mathcal{F}(\mathcal{A})$ be a facet of codimension 1. There are exactly two chambers $C, D \in \mathcal{C}(\mathcal{A})$ such that $F \leq C$ and $F \leq D$. Then there are two edges, $U(F, C)$ and $U(F, D)$, joining $\omega(C)$ and $\omega(D)$ in the 1-skeleton of $\text{Sal}(\mathcal{A})$ (see Figure 5.3). We use the convention that $U(F, C)$ is endowed with an orientation which goes from $\omega(C)$ to $\omega(D)$.

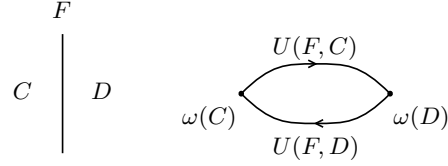


Figure 5.3. 1-skeleton of $\text{Sal}(\mathcal{A})$.

2-skeleton. Let $F_b \in \mathcal{F}(\mathcal{A})$ be a facet of codimension 2, and let $C_b \in \mathcal{C}(\mathcal{A})$ such that $F_b \leq C_b$. Let $C_0 = D_0 = C_b, C_1, \dots, C_l = D_l, \dots, D_1$ be the chambers $C \in \mathcal{C}(\mathcal{A})$ such that $F_b \leq C$, arranged like in Figure 5.4. Let $F_1, \dots, F_l, G_1, \dots, G_l$ be the facets $F \in \mathcal{F}(\mathcal{A})$ of codimension 1 such that $F_b \leq F$, arranged like in Figure 5.4. Set $a_i = U(F_i, C_{i-1})$ and $b_i = U(G_i, D_{i-1})$ for $1 \leq i \leq l$. Then $U(F_b, C_b)$ is a 2-disk whose boundary is $(a_1 a_2 \dots a_l)(b_1 b_2 \dots b_l)^{-1}$.

Let Γ be a Coxeter graph of spherical type, let (W_Γ, S) be the Coxeter system of type Γ , and let (G_Γ, Σ) be the Artin system of type Γ . Recall the set $\Pi = \{e_s; s \in S\}$ of simple roots, the linear space $V = \oplus_{s \in S} \mathbb{R}e_s$, and the canonical bilinear form $\langle, \rangle : V \times V \rightarrow \mathbb{R}$. Recall also from Theorem 3.8 that \langle, \rangle is positive definite, and that $W = W_\Gamma$ can be viewed as a finite subgroup of $O(V) = O(V, \langle, \rangle)$ generated by reflections.

Let \mathcal{A}_Γ denote the set of reflecting hyperplanes of W . Then $M_\Gamma = M(\mathcal{A}_\Gamma)$, the group W_Γ acts freely on M_Γ , $N_\Gamma = M_\Gamma/W_\Gamma$, and $\pi_1(N_\Gamma) = G_\Gamma$ (see Subsection 3.4).

Fix a (base) chamber $C_b \in \mathcal{C}(\mathcal{A}_\Gamma)$. A hyperplane $H \in \mathcal{A}_\Gamma$ is called a *wall* of C_b if $\text{codim}(\bar{C}_b \cap H) = 1$. The following is proved in [28].

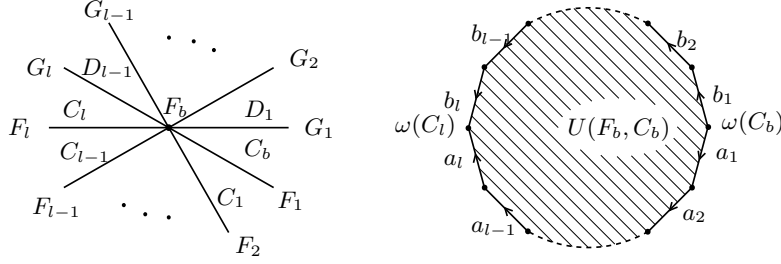


Figure 5.4. 2-skeleton of $\text{Sal}(\mathcal{A})$.

Proposition 5.9. (1) C_b is a simplicial cone.

(2) Let H_1, \dots, H_n be the walls of C_b , and, for $1 \leq i \leq n$, let s_i be the orthogonal reflection with respect to H_i . Then, up to conjugation, $S = \{s_1, \dots, s_n\}$ is the Coxeter generating set of W .

For $T \subset S$ we denote by W_T the subgroup of W generated by T , and by Γ_T the full subgraph of Γ spanned by T . It is a well-know fact (see [28], for example) that (W_T, T) is the Coxeter system of type Γ_T . The *Coxeter complex* of (W, S) is defined to be the set

$$\text{Cox}_\Gamma = \{wW_T ; T \subset S \text{ and } w \in W\}$$

ordered by the reverse inclusion (*i.e.* $w_1W_{T_1} \leq w_2W_{T_2}$ if $w_1W_{T_1} \supset w_2W_{T_2}$).

We fix a base chamber C_b and we take $S = \{s_1, \dots, s_n\}$ like in Proposition 5.9. For each $s \in S$ we denote by H_s the hyperplane fixed by s . So, $\{H_s; s \in S\}$ is the set of walls of C_b . Since C_b is a simplicial cone, for every $T \subset S$ there exists a unique facet $F(T) \in \mathcal{F}(\mathcal{A}_\Gamma)$ such that $F(T) \leq C_b$ and $\langle F(T) \rangle = \cap_{s \in T} H_s$. The proof of the following can be found in [28].

Proposition 5.10. The map

$$\begin{aligned} \psi : \text{Cox}_\Gamma &\rightarrow \mathcal{F}(\mathcal{A}_\Gamma) \\ wW_T &\mapsto wF(T) \end{aligned}$$

is well-defined and is an isomorphism of ordered sets.

Now, the following lemmas 5.11 and 5.12 are used to describe the poset \mathcal{X} in terms of Coxeter complexes.

Lemma 5.11 (Bourbaki [28]). Let $T \subset S$ and $w \in W$. Then wW_T has a smallest element u for the order \leq_L (defined in Subsection 3.2). That is, for all $w' \in wW_T$ there exists a unique $v' \in W_T$ such that $w' = uv'$ and $\text{lg}_S(w') = \text{lg}_S(u) + \text{lg}_S(v')$.

The smallest element of wW_T is denoted by $u = \min_T(w)$, and such an element is called *T-minimal*. The set of *T*-minimal elements is denoted by $\text{Min}(T)$. For $w \in W$, we denote by $\pi_T(w)$ the element $v \in W_T$ such that $w = \min_T(w) \cdot v$.

The proof of the following is left to the reader.

Lemma 5.12. *Let C_b be a base chamber, let $T \subset S$, and let $F = F(T)$. Let $w_1, w_2 \in W$. We have $(w_1 C_b)_F = (w_2 C_b)_F$ if and only if $\pi_T(w_1) = \pi_T(w_2)$.*

Set

$$\widehat{\text{Cox}}_\Gamma = \{(T, w) ; w \in W \text{ and } T \subset S\}.$$

Let \leq be the partial order on $\widehat{\text{Cox}}_\Gamma$ defined by

$$(T_1, w_1) \leq (T_2, w_2) \quad \text{if} \quad T_1 \supset T_2, \quad \min_{T_1}(w_1) = \min_{T_1}(w_2), \\ \text{and} \quad \pi_{T_2}(w_1) = \pi_{T_2}(w_2).$$

Note that the conditions “ $T_1 \supset T_2$ and $\min_{T_1}(w_1) = \min_{T_1}(w_2)$ ” are equivalent to the condition $w_1 W_{T_1} \supset w_2 W_{T_2}$, and, by Lemma 5.12, the condition $\pi_{T_2}(w_1) = \pi_{T_2}(w_2)$ is equivalent to the condition $(w_1 C_b)_{F(T_2)} = (w_2 C_b)_{F(T_2)}$. So:

Theorem 5.13. *The map*

$$\begin{aligned} \hat{\psi} : \widehat{\text{Cox}}_\Gamma &\rightarrow \mathcal{X}(\mathcal{A}_\Gamma) \\ (T, w) &\mapsto (wF(T), wC_b) \end{aligned}$$

is well-defined and is an isomorphism of posets.

For $(T, w) \in \widehat{\text{Cox}}_\Gamma$ we set $U(T, w) = U(\hat{\psi}(T, w))$. So, $\{U(T, w); (T, w) \in \widehat{\text{Cox}}_\Gamma\}$ is a cellular decomposition of $\text{Sal}(\mathcal{A}_\Gamma)$. Moreover, the dimension of $U(T, w)$ is $|T|$ for all $(T, w) \in \widehat{\text{Cox}}_\Gamma$.

The Coxeter group W acts on $\widehat{\text{Cox}}_\Gamma$ by

$$u \cdot (T, w) = (T, uw), \quad \text{for } (T, w) \in \widehat{\text{Cox}}_\Gamma \text{ and } u \in W.$$

It turns out that this action preserves the order of $\widehat{\text{Cox}}_\Gamma$ and induces a cellular action on $\text{Sal}(\mathcal{A}_\Gamma)$ defined by

$$u \cdot U(T, w) = U(T, uw) \quad \text{for } (T, w) \in \widehat{\text{Cox}}_\Gamma \text{ and } u \in W.$$

Theorem 5.14 (Salvetti [141]). *There exists an embedding $\text{Sal}(\mathcal{A}_\Gamma) \hookrightarrow M_\Gamma$ and a (strong) retracting deformation of M_Γ onto $\text{Sal}(\mathcal{A}_\Gamma)$ that are equivariant under the action of W . In particular, there exists an embedding $\text{Sal}(\mathcal{A}_\Gamma)/W \hookrightarrow M_\Gamma/W = N_\Gamma$ and a (strong) retracting deformation of N_Γ onto $\text{Sal}(\mathcal{A}_\Gamma)/W$.*

To each $T \subset S$ corresponds a unique cell $U_N(T)$ of $\text{Sal}(\mathcal{A}_\Gamma)/W$ of dimension $|T|$. This cell is the orbit of $U(T, w)$ for all $w \in W$. Every cell of $\text{Sal}(\mathcal{A}_\Gamma)/W$ is of this form.

The 0-skeleton of $\text{Sal}(\mathcal{A}_\Gamma)/W$ contains a unique vertex, $\omega_N = U_N(\emptyset)$. For every $s \in S$ there is an edge $U_N(s)$ in $\text{Sal}(\mathcal{A}_\Gamma)/W$ and each edge is of this form. For every pair $\{s, t\} \subset S$ there is a 2-cell $U_N(s, t)$ in $\text{Sal}(\mathcal{A}_\Gamma)/W$ whose boundary is

$$\text{prod}(U_N(s), U_N(t) : m_{st}) \cdot \text{prod}(U_N(t), U_N(s) : m_{st})^{-1},$$

and every 2-cell is of this form. Note that the 2-skeleton of $\text{Sal}(\mathcal{A}_\Gamma)/W$ is equal to the 2-cell complex associated to the standard presentation of G_Γ . This gives an alternative proof to Theorems 2.2 and 3.14.

For $0 \leq q \leq |S|$, set

$$C_q(G_\Gamma) = \bigoplus_{\substack{T \subset S \\ |T|=q}} \mathbb{Z}[G_\Gamma] \cdot E_T,$$

the free $\mathbb{Z}[G_\Gamma]$ -module freely spanned by $\{E_T; T \subset S \text{ and } |T| = q\}$. We fix a total order $S = \{s_1, \dots, s_n\}$ on S and we define $d : C_q(G_\Gamma) \rightarrow C_{q-1}(G_\Gamma)$ as follows. Let $T = \{s_{i_1}, \dots, s_{i_q}\} \subset S$, $i_1 < \dots < i_q$. Then

$$dE_T = \sum_{j=1}^q (-1)^{j-1} \left(\sum_{\substack{u \in W_T \\ u \in \text{Min}(T \setminus \{s_{i_j}\})}} (-1)^{\text{lg}_S(u)} \kappa(u) \right) \cdot E_{T \setminus \{s_{i_j}\}},$$

where $\kappa : W \rightarrow G_\Gamma$ is the set-section of the canonical epimorphism $\theta : G_\Gamma \rightarrow W$ defined in Subsection 3.3.

Theorem 5.15 (De Concini, Salvetti [63], Squier [144]). *The complex $(C_*(G_\Gamma), d)$ is a free resolution of \mathbb{Z} by $\mathbb{Z}[G_\Gamma]$ -modules.*

Note. Squier's proof of Theorem 5.15 does not use the Salvetti complexes at all and is independent from the proof of De Concini and Salvetti.

6 Linear representations

The existence (or non-existence) of faithful linear representations of the braid groups was one of the major problem in the field. This problem was solved by Bigelow [17] and Krammer [106] in 2000. Their representation, which is known now as the LKB representation, was right afterwards extended to the Artin groups of type D_n ($n \geq 4$) and E_k ($k = 6, 7, 8$) by Digne [74], Cohen, and

Wales [51], and to all Artin groups of small type in [132]. The representations of Digne, Cohen and Wales are proved to be faithful. Hence, since any spherical type Artin group embeds in a direct product of Artin groups of type A_n ($n \geq 1$), D_n ($n \geq 4$), and E_k ($k = 6, 7, 8$) (see [57]), any Artin group of spherical type is linear. The extension to the non-spherical type Artin groups gives rise to a linear representation over an infinite dimensional vector space, so it cannot be used for proving that these groups are linear. However, these representations are useful tools to study the non-spherical type Artin groups. In particular, they are the main tool in the proof of Theorem 3.9.

In Subsection 6.1 we present the algebraic approach to the LKB representations as constructed in [132] for the Artin groups of small type. Subsection 6.2 is dedicated to the topological construction of the LKB representations. Curiously, this topological point of view is known only for the braid groups.

6.1 Algebraic approach

Let Γ be a Coxeter graph, let $M = (m_{st})_{s,t \in S}$ be the Coxeter matrix of Γ , let (W_Γ, S) be the Coxeter system of type Γ , let (G_Γ, Σ) be the Artin system of type Γ , and let G_Γ^+ be the Artin monoid of type Γ .

We say that Γ is of *small type* if $m_{st} \leq 3$ for all $s, t \in S$, $s \neq t$, and we say that Γ is *without triangle* if there is no triple $\{s, t, r\}$ in S such that $m_{st} = m_{tr} = m_{rs} = 3$. We assume from now on that Γ is of small type and without triangle.

Recall from Subsection 3.2 the set $\Pi = \{e_s; s \in S\}$ of simple roots, the space $V = \bigoplus_{s \in S} \mathbb{R}e_s$, the canonical bilinear form $\langle, \rangle : V \times V \rightarrow \mathbb{R}$, and the root system $\Phi = \{we_s; s \in S \text{ and } w \in W\}$. Recall also that we have the disjoint union $\Phi = \Phi_+ \sqcup \Phi_-$, where Φ_+ is the set of positive roots and Φ_- is the set of negative roots (see Proposition 3.4).

Set $\mathcal{E} = \{u_f; f \in \Phi_+\}$ an abstract set in one-to-one correspondence with Φ_+ , and $\mathbb{K} = \mathbb{Q}(x, y)$. Note that \mathcal{E} is finite if and only if Γ is of spherical type. We denote by \mathcal{V} the \mathbb{K} -vector space having \mathcal{E} as a basis.

For all $s \in S$ we define a linear transformation $\varphi_s : \mathcal{V} \rightarrow \mathcal{V}$ by

$$\varphi_s(u_f) = \begin{cases} 0 & \text{if } f = e_s \\ u_f & \text{if } \langle e_s, f \rangle = 0 \\ y \cdot u_{f-ae_s} & \text{if } \langle e_s, f \rangle = a > 0 \text{ and } f \neq e_s \\ (1-y) \cdot u_f + u_{f+ae_s} & \text{if } \langle e_s, f \rangle = -a < 0 \end{cases}$$

The following is easy to prove.

Lemma 6.1. *The mapping $\sigma_s \mapsto \varphi_s$, $s \in S$, induces a homomorphism of monoids $\varphi : G_\Gamma^+ \rightarrow \text{End}(\mathcal{V})$.*

For all $s \in S$ and all $f \in \Phi_+$ we choose a polynomial $T(s, f) \in \mathbb{Q}[y]$ and we define $\Phi_s : \mathcal{V} \rightarrow \mathcal{V}$ by

$$\Phi_s(u_f) = \varphi_s(u_f) + x \cdot T(s, f) \cdot u_{e_s}.$$

Now, we have:

Theorem 6.2 (Paris [132]). *There exists a choice of polynomials $T(s, f)$, $s \in S$ and $f \in \Phi_+$, such that the mapping $\sigma_s \mapsto \Phi_s$, $s \in S$, induces a homomorphism $\Phi : G_\Gamma^+ \rightarrow \text{GL}(\mathcal{V})$.*

Theorem 6.3 (Paris [132]). *The above defined homomorphism $\Phi : G_\Gamma^+ \rightarrow \text{GL}(\mathcal{V})$ is injective.*

Corollary 6.4 (Paris [132]). *The natural homomorphism $\iota : G_\Gamma^+ \rightarrow G_\Gamma$ is injective.*

Proof. Since G_Γ is the group of fractions of G_Γ^+ , there exists a unique homomorphism $\hat{\Phi} : G_\Gamma \rightarrow \text{GL}(\mathcal{V})$ such that $\Phi = \hat{\Phi} \circ \iota$. Since Φ is injective, we conclude that ι is also injective. \square

Corollary 6.5 (Bigelow [17], Krammer [106], Digne [74], Cohen, Wales [51]). *Suppose that Γ is of spherical type. Let $\hat{\Phi} : G_\Gamma \rightarrow \text{GL}(\mathcal{V})$ be the homomorphism induced by Φ . Then $\hat{\Phi}$ is injective.*

Proof. Let $\alpha \in \text{Ker } \hat{\Phi}$. By Proposition 4.8, α can be written in the form $\alpha = \beta^{-1}\gamma$, with $\beta, \gamma \in G_\Gamma^+$. We have $1 = \hat{\Phi}(\alpha) = \Phi(\beta)^{-1}\Phi(\gamma)$, thus $\Phi(\beta) = \Phi(\gamma)$. Since Φ is injective, it follows that $\beta = \gamma$, thus $\alpha = \beta^{-1}\gamma = 1$. \square

Note. It is shown in [132] that any Artin monoid G_Γ^+ can be embedded in an Artin monoid G_Ω^+ , where Ω is of small type without triangle. Moreover, if Γ is of spherical type, then Ω can be chosen to be of spherical type (see also [57], [90], [58], [41]). So, Corollary 6.4 implies that $\iota : G_\Gamma^+ \rightarrow G_\Gamma$ is injective for all Coxeter graphs Γ , and Corollary 6.5 implies that all the Artin groups of spherical type are linear.

Note. It is shown in [120] that: if Γ is of type A_n, D_n, E_k ($k = 6, 7, 8$), then the image of $\hat{\Phi}$ is Zariski dense in $\text{GL}(\mathcal{V})$. In particular, this shows that $\hat{\Phi}$ is irreducible (see also [154], [121], [50]).

Note. The proof of Theorem 6.3 given in [132] is largely inspired by Krammer's proof of the same theorem for the braid groups [106]. A new, short, and elegant proof can be found now in [95].

6.2 Topological approach

Now, we give a topological interpretation of the representation $\hat{\Phi} : G_\Gamma \rightarrow \text{GL}(\mathcal{V})$ in the case $\Gamma = A_{n-1}$, that is, when $G_\Gamma = \mathcal{B}_n$ is the braid group on n strands. Such an interpretation is unknown for the other Artin groups.

Let M be a connected CW-complex, let $G = \pi_1(M)$, and let R be a (right) $\mathbb{Z}[G]$ -module. Let \tilde{M} be the universal cover of M . The action of G on \tilde{M} induces an action of G on the group $C_q(\tilde{M})$ of (cellular) q -chains of \tilde{M} , and this action makes $C_q(\tilde{M})$ a module over the group ring $\mathbb{Z}[G]$. It is also easily seen that the boundary maps $\partial : C_q(\tilde{M}) \rightarrow C_{q-1}(\tilde{M})$ are $\mathbb{Z}[G]$ -module homomorphisms. We define $C_q(M, R)$ to be $R \otimes_{\mathbb{Z}[G]} C_q(\tilde{M})$. These groups form a chain complex with boundary map $\text{Id} \otimes \partial$. The homology groups $H_q(M, R)$ of this chain complex are the *homology groups of M with local coefficients R* .

Now, for $n \geq 1$, M_n denotes the space of ordered configurations of n points in \mathbb{C} , and $N_n = M_n / \text{Sym}_n$ denotes the space of (unordered) configurations of n points in \mathbb{C} (see Section 2). Let $n, m \geq 2$. By [80], the map

$$\begin{aligned} p_{n,m} : \quad M_{n+m} &\rightarrow M_n \\ (z_1, \dots, z_n, z_{n+1}, \dots, z_{n+m}) &\mapsto (z_1, \dots, z_n) \end{aligned}$$

is a locally trivial fiber bundle which admits a cross-section. The fiber of $p_{n,m}$ is as follows. Set

$$\begin{aligned} H_{ij} &= \{w \in \mathbb{C}^m ; w_i = w_j\} && \text{for } 1 \leq i < j \leq m \\ K_{ik} &= \{w \in \mathbb{C}^m ; w_i = k\} && \text{for } 1 \leq i \leq m \text{ and } 1 \leq k \leq n \end{aligned}$$

Set

$$X_{n,m} = \mathbb{C}^m \setminus \left(\left(\bigcup_{i < j} H_{ij} \right) \cup \left(\bigcup_{\substack{1 \leq i \leq m \\ 1 \leq k \leq n}} K_{ik} \right) \right).$$

Then

$$p_{n,m}^{-1}(1, 2, \dots, n) = \{(1, 2, \dots, n)\} \times X_{n,m}.$$

Let $\text{Sym}_n \times \text{Sym}_m$ act on M_{n+m} , Sym_n acting by permutations on the n first coordinates, and Sym_m acting on the m last ones. Set

$$\begin{aligned} N_{n,m} &= M_{n+m} / (\text{Sym}_n \times \text{Sym}_m), \\ Y_{n,m} &= X_{n,m} / \text{Sym}_m. \end{aligned}$$

Then $p_{n,m}$ induces a locally trivial fiber bundle $\bar{p}_{n,m} : N_{n,m} \rightarrow N_n$ whose fiber is $Y_{n,m}$.

For $z \in \mathbb{C}^n$ we set

$$\|z\|_\infty = \max\{|z_i| ; 1 \leq i \leq n\}.$$

It is easily checked that the map

$$\begin{aligned} \kappa : M_n &\rightarrow M_{n+m} \\ z &\mapsto (z, \|z\|_\infty + 1, \|z\|_\infty + 2, \dots, \|z\|_\infty + m) \end{aligned}$$

is a well-defined cross-section of $p_{n,m}$ which is equivariant by the action of Sym_n , thus it induces a cross-section $\bar{\kappa} : N_n \rightarrow N_{n,m}$ of $\bar{p}_{n,m}$. By the homotopy long exact sequence of a fiber bundle (see Theorem 2.9), we conclude that $\pi_1(N_{n,m})$ can be written as a semi-direct product $\pi_1(N_{n,m}) = \pi_1(Y_{n,m}) \rtimes \mathcal{B}_n$.

Set $G_{n,m} = \pi_1(Y_{n,m})$. We consider $G_{n,m}$ as a subgroup of $\pi_1(N_{n,m})$ which, in its turn, is viewed as a subgroup of $\pi_1(N_{n+m}) = \mathcal{B}_{n+m}$. It is easily seen that $G_{n,m}$ is generated by the set

$$\{\sigma_k ; n+1 \leq k \leq n+m\} \cup \{\delta_{ik} ; 1 \leq i \leq n \text{ and } n+1 \leq k \leq n+m\},$$

where δ_{ik} is the pure braid defined in Theorem 2.3. Let b be the homology class of σ_{n+1} in $H_1(G_{n,m}) = H_1(Y_{n,m})$, and let a_i be the homology class of $\delta_{i,n+1}$, $1 \leq i \leq n$. The proof of the following is left to the reader.

Proposition 6.6. *$H_1(Y_{n,m}) = H_1(G_{n,m})$ is a free abelian group freely generated by $\{b, a_1, a_2, \dots, a_n\}$.*

Let $\bar{\rho} : H_1(G_{n,m}) \rightarrow \mathbb{Q}(x, y)^*$ be the homomorphism which sends a_i to x for all $1 \leq i \leq n$, and sends b to y . Let $\rho : G_{n,m} \rightarrow \mathbb{Q}(x, y)^*$ be the composition of the natural projection $G_{n,m} \rightarrow H_1(G_{n,m})$ with $\bar{\rho}$. This homomorphism makes $\mathbb{Q}(x, y)$ a $\mathbb{Z}[G_{n,m}]$ -module that we denote by Γ_ρ .

The proof of the following is also left to the reader.

Proposition 6.7. *The kernel of ρ is invariant under the action of \mathcal{B}_n , and \mathcal{B}_n acts trivially on the quotient $G_{n,m}/\text{Ker } \rho \simeq \mathbb{Z} \times \mathbb{Z}$.*

From Proposition 6.7 follows that the fibration $\bar{p}_{n,m} : N_{n,m} \rightarrow N_n$ induces a monodromy representation $\Phi_{n,m} : \mathcal{B}_n \rightarrow \text{Aut}_{\mathbb{Q}(x,y)}(H_*(Y_{n,m}, \Gamma_\rho))$.

The following was announced by Krammer [105], [106], and proved in [17] (see also [129]).

Theorem 6.8 (Bigelow [17]). *The homomorphism $\Phi_{n,2} : \mathcal{B}_n \rightarrow \text{Aut}_{\mathbb{Q}(x,y)}(H_2(Y_{n,2}, \Gamma_\rho))$ coincides with the representation $\hat{\Phi} : G_{A_{n-1}} \rightarrow \text{GL}(\mathcal{V})$ defined in Subsection 6.1.*

Note. The representation $\hat{\Phi} : G_{A_{n-1}} \rightarrow \text{GL}(\mathcal{V})$ also coincides with the representation studied by Lawrence in [112]. Lawrence's construction is also geometric. It slightly differs from the one presented above, but I do not know exactly how to relate them without the formulas.

Note. It is announced in [152] that $\Phi_{n,m}$ is faithful for all $m \geq 2$, and it is announced in [48] that $\Phi_{n,m} : \mathcal{B}_n \rightarrow \text{Aut}_{\mathbb{Q}(x,y)}(H_m(Y_{n,m}, \Gamma_\rho))$ is irreducible for all $m \geq 2$.

7 Geometric representations

7.1 Definitions and examples

Let Σ be an oriented compact surface, possibly with boundary, and let \mathcal{P} be a finite collection of punctures in the interior of Σ . Let $\mathcal{M}(\Sigma, \mathcal{P})$ denote the mapping class group of the pair (Σ, \mathcal{P}) , as defined in Subsection 2.3. Let Γ be a Coxeter graph, and let G_Γ be the Artin group of type Γ . Define a *geometric representation* of G_Γ in $\mathcal{M}(\Sigma, \mathcal{P})$ to be a homomorphism from G_Γ to $\mathcal{M}(\Sigma, \mathcal{P})$.

The main tools for constructing geometric representations of Artin groups are the Dehn twists and the braid twists. The braid twists are defined in Subsection 2.3, and the Dehn twists are defined as follows.

An *essential circle* is an embedding $a : \mathbb{S}^1 \hookrightarrow \Sigma \setminus \mathcal{P}$ of the circle whose image is contained in the interior of Σ and does not bound any disk in Σ containing 0 or 1 puncture. Two essential circles a, a' are *isotopic* if there exists a continuous family $\{a_t\}_{t \in [0,1]}$ of essential circles such that $a = a_0$ and $a' = a_1$. Isotopy of essential circles is an equivalence relation that we denote by $a \sim a'$.

Let $a : \mathbb{S}^1 \rightarrow \Sigma \setminus \mathcal{P}$ be an essential circle. Take an embedding $A : [0, 1] \times \mathbb{S}^1 \rightarrow \Sigma \setminus \mathcal{P}$ of the annulus such that $A(\frac{1}{2}, z) = a(z)$ for all $z \in \mathbb{S}^1$, and define $T \in \text{Homeo}^+(\Sigma, \mathcal{P})$ by

$$(T \circ A)(t, z) = A(t, e^{2i\pi t} z),$$

and T is the identity outside the image of A (see Figure 7.1). The *Dehn twist* along a , denoted by σ_a , is defined to be the element of $\mathcal{M}(\Sigma, \mathcal{P})$ represented by T . Note that

- the definition of σ_a does not depend on the choice of the map A ;
- if a is isotopic to a' , then $\sigma_a = \sigma_{a'}$.

Recall that, for an essential arc a of (Σ, \mathcal{P}) , τ_a denotes the braid twist along a . The Dehn twists and the braid twists satisfy the following relations (see [20], [110]).

Proposition 7.1. (1) *Let a, b be two essential circles that intersect transversely. Then*

$$\begin{aligned} \sigma_a \sigma_b &= \sigma_b \sigma_a & \text{if } a \cap b = \emptyset \\ \sigma_a \sigma_b \sigma_a &= \sigma_b \sigma_a \sigma_b & \text{if } |a \cap b| = 1 \end{aligned}$$

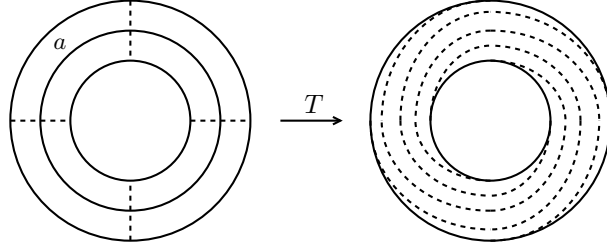


Figure 7.1. Dehn twist.

(2) Let a, b be two essential arcs of (Σ, \mathcal{P}) . Then

$$\begin{array}{ll} \tau_a \tau_b = \tau_b \tau_a & \text{if } a \cap b = \emptyset \\ \tau_a \tau_b \tau_a = \tau_b \tau_a \tau_b & \text{if } a(0) = b(1) \text{ and } a \cap b = \{a(0)\} \end{array}$$

(3) let a be an essential arc, and let b be an essential circle which intersects a transversely. Then

$$\begin{array}{ll} \tau_a \sigma_b = \sigma_b \tau_a & \text{if } a \cap b = \emptyset \\ \tau_a \sigma_b \tau_a \sigma_b = \sigma_b \tau_a \sigma_b \tau_a & \text{if } |a \cap b| = 1 \end{array}$$

Example 1. Suppose $\Sigma = \mathbb{D}$ is a disk, and $\mathcal{P}_n = \{P_1, \dots, P_n\}$ is a collection of n punctures in the interior of Σ . Then the Artin isomorphism $\Phi : \mathcal{B}_n \rightarrow \mathcal{M}(\mathbb{D}, \mathcal{P}_n)$ of Theorem 2.16 is a geometric representation of $G_{A_{n-1}} = \mathcal{B}_n$.

Example 2. Let $n \geq 3$. Suppose that, if n is odd, then Σ is a surface of genus $\frac{n-1}{2}$ with one boundary component, and if n is even, then Σ is a surface of genus $\frac{n-2}{2}$ with two boundary components. Let a_1, \dots, a_{n-1} be the essential circles of Σ pictured in Figure 7.2. By Proposition 7.1, the mapping $\sigma_i \mapsto \sigma_{a_i}$, $1 \leq i \leq n-1$, induces a representation $\rho_M : \mathcal{B}_n \rightarrow \mathcal{M}(\Sigma)$ called the *monodromy representation* of \mathcal{B}_n . This geometric representation was introduced by Birman and Hilden in [25], where it is proved that ρ_M is faithful and its image consists on mapping classes arising from homeomorphisms symmetric with respect to a hyperelliptic involution (see also [26], [153], and [117]). It is also the geometric monodromy of the simple singularity of type A_{n-1} (see [134]). Let $P_0 \in \partial\Sigma$ be a base-point. Then ρ_M induces a homomorphism $\rho_{M*} : \mathcal{B}_n \rightarrow \text{Aut}(\pi_1(\Sigma, P_0))$ which turns out to coincide with the homomorphism $\rho_D : \mathcal{B}_n \rightarrow \text{Aut}(F_{n-1})$ defined in Subsection 3.1 (see [60]).

Example 3. Let $\mathbb{D}^2 = \{z \in \mathbb{C}; |z| \leq 1\}$ be the standard disk. A *chord diagram* in \mathbb{D}^2 is defined to be a collection $\{S_1, \dots, S_n\}$ of segments in \mathbb{D}^2 such that

- the extremities of S_i belong to $\partial\mathbb{D}^2$ and its interior is contained in the interior of \mathbb{D}^2 , for all $1 \leq i \leq n$;

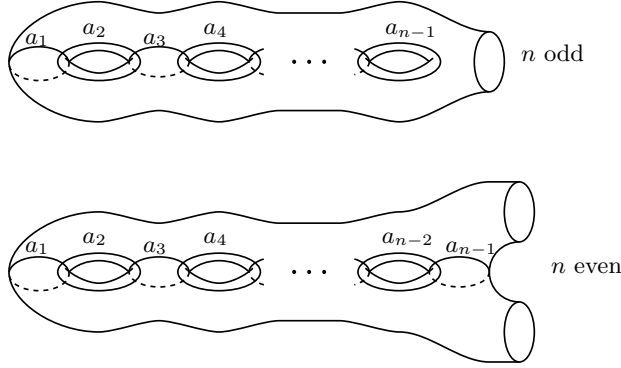


Figure 7.2. Monodromy representation of \mathcal{B}_n .

- either S_i and S_j are disjoint, or they intersect transversely in a unique point in the interior of \mathbb{D}^2 , for all $1 \leq i \neq j \leq n$.

From this data one can define a Coxeter matrix $M = (m_{ij})_{1 \leq i, j \leq n}$ setting $m_{ij} = 2$ if S_i and S_j are disjoint, and $m_{ij} = 3$ if they intersect. The Coxeter graph Γ of M is called the *intersection diagram* of the chord diagram.

From this data one can also define a surface Σ by attaching to \mathbb{D}^2 a handle H_i which joins both extremities of S_i , for all $1 \leq i \leq n$ (see Figure 7.3). Let a_i be the essential circle of Σ made with S_i and the central arc of H_i . Then, by Proposition 7.1, the mapping $\sigma_i \mapsto \sigma_{a_i}$, $1 \leq i \leq n$, induces a geometric representation $\rho_{PV} : G_\Gamma \rightarrow \mathcal{M}(\Sigma)$, called *Perron-Vannier representation*.

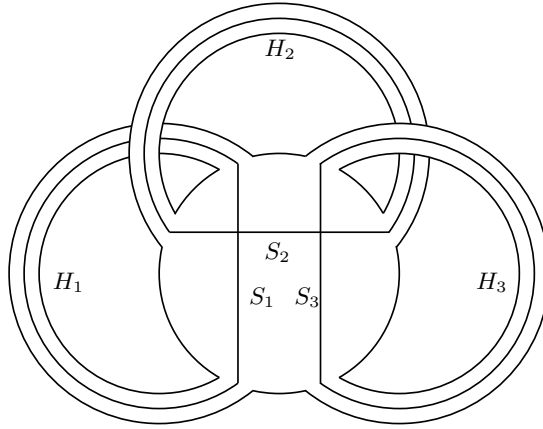


Figure 7.3. Chord diagram and associated surface.

The Perron-Vannier representations were introduced in [134]. If $\Gamma = A_{n-1}$, then ρ_{PV} is equal to the monodromy representation ρ_M defined in Example 2.

More generally, if Γ is A_n ($n \geq 1$), D_n ($n \geq 4$), or E_k ($k = 6, 7, 8$), then ρ_{PV} is the geometric monodromy of the simple singularity of type Γ (see [134]). For a connected graph Γ , the representation ρ_{PV} is faithful if and only if either $\Gamma = A_n$ for some $n \geq 1$, or $\Gamma = D_n$ for some $n \geq 4$ (see [134], [109], [151]).

Example 4. This example comes from [58]. Recall that a Coxeter graph Γ is of *small type* if $m_{st} \leq 3$ for all $s, t \in S$, where $M = (m_{st})_{s,t \in S}$ is the Coxeter matrix of Γ . Let Γ be a small type Coxeter graph. We choose (arbitrarily) a total order $<$ on S . For $s \in S$, we set $St_s = \{t \in S; m_{st} = 3\} \cup \{s\}$. Write $St_s = \{t_1, t_2, \dots, t_k\}$ such that $t_1 < t_2 < \dots < t_k$, and suppose that $s = t_j$. For $1 \leq i \leq k$, the difference $i - j$ is called the *relative position* of t_i with respect to s and is denoted by $\text{pos}(t_i : s)$. In particular, $\text{pos}(s : s) = 0$.

Let $s \in S$ and let $k = |St_s|$. Let An_s denote the annulus $An_s = (\mathbb{R}/2k\mathbb{Z}) \times [0, 1]$. We define the surface $\Sigma = \Sigma_\Gamma$ by

$$\Sigma = \left(\bigsqcup_{s \in S} An_s \right) / \sim,$$

where \sim is the equivalence relation defined as follows. Let $s, t \in S$ such that $s < t$ and $m_{st} = 3$. Set $p = \text{pos}(t : s) > 0$ and $q = \text{pos}(s : t) < 0$. For all $(x, y) \in [0, 1] \times [0, 1]$ the relation \sim identifies the point $(2p + x, y)$ of An_s with the point $(2q + 1 - y, x)$ of An_t (see Figure 7.4).

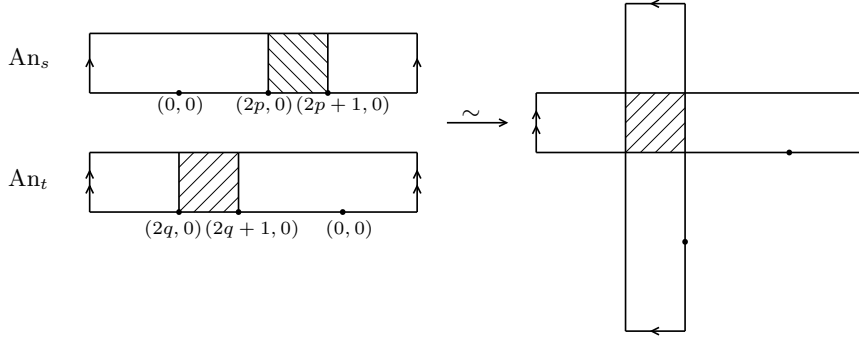


Figure 7.4. Identification of annuli.

We identify each annulus An_s with its image in Σ , and we denote by a_s its central curve. Note that a_s is an essential circle, $a_s \cap a_t = \emptyset$ if $m_{st} = 2$, and $|a_s \cap a_t| = 1$ if $m_{st} = 3$. So, by Proposition 7.1, the mapping $\sigma_s \mapsto \sigma_{a_s}$, $s \in S$, induces a geometric representation $\rho_{CP} : G_\Gamma \rightarrow \mathcal{M}(\Sigma)$.

We have $\rho_{CP} = \rho_{PV}$ if Γ is a tree. (Note that it may happen that ρ_{PV} is not defined if Γ is not a tree.) If $\Gamma = \tilde{A}_n$, then ρ_{CP} is faithful (while, by [109], ρ_{PV} is not faithful in this case).

7.2 Presentations

Let $\Sigma_{g,r}$ be a surface of genus $g \geq 1$ with $r \geq 0$ boundary components, and let \mathcal{P}_n be a collection of n punctures in the interior of $\Sigma_{g,r}$, where $n \geq 0$.

Assume first that $r \geq 1$. Consider the essential circles $a_0, a_1, \dots, a_r, b_1, b_2, \dots, b_{2g-1}, c, d_1, \dots, d_{r-1}$, and the essential arcs e_1, e_2, \dots, e_{n-1} drawn in Figure 7.5. Note that there is no c if $g = 1$, there is no d_i if $r = 1$, there is no a_r if $n = 0$, and there is no e_i if $n = 0$ or 1 . Let $\Gamma(g, r, n)$ be the Coxeter graph drawn in Figure 7.6. One can show that the set

$$\{\sigma_{a_0}, \sigma_{a_1}, \dots, \sigma_{a_r}, \sigma_{b_1}, \sigma_{b_2}, \dots, \sigma_{b_{2g-1}}, \sigma_c, \sigma_{d_1}, \sigma_{d_2}, \dots, \sigma_{d_{r-1}}, \tau_{e_1}, \dots, \tau_{e_{n-1}}\}$$

generates $\mathcal{M}(\Sigma_{g,r}, \mathcal{P}_n)$. On the other hand, by Proposition 7.1, the mapping

$$\begin{aligned} x_i &\mapsto \sigma_{a_i} \quad (0 \leq i \leq r), & y_i &\mapsto \sigma_{b_i} \quad (1 \leq i \leq 2g-1), & z &\mapsto \sigma_c \\ u_i &\mapsto \sigma_{d_i} \quad (1 \leq i \leq r-1), & v_j &\mapsto \tau_{e_j} \quad (1 \leq j \leq n-1), \end{aligned}$$

induces a homomorphism $\rho : G_{\Gamma(g,r,n)} \rightarrow \mathcal{M}(\Sigma_{g,r}, \mathcal{P}_n)$. So, in order to obtain a presentation for $\mathcal{M}(\Sigma_{g,r}, \mathcal{P}_n)$, it suffices to find normal generators for $\text{Ker } \rho$. This was done in [124] for $r = 1$ and $n = 0$, and in [110] for the other cases.

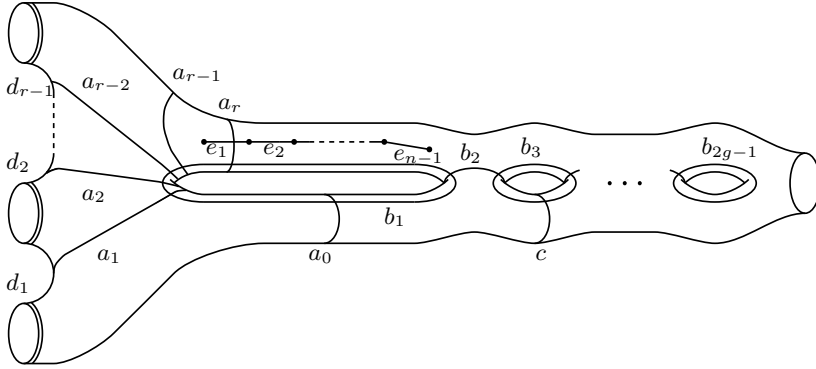


Figure 7.5. Generators of $\mathcal{M}(\Sigma_{g,r}, \mathcal{P}_n)$.

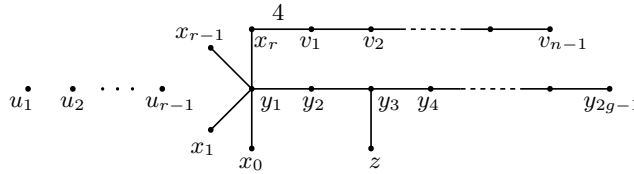


Figure 7.6. The Coxeter graph $\Gamma(g, r, n)$.

One can use the same kind of arguments for the case $r = 0$. Consider the essential circles $a_0, a_1, b_1, b_2, \dots, b_{2g-1}, c$, and the essential arcs e_1, e_2, \dots, e_{n-1} drawn in Figure 7.7. Then the set

$$\{\sigma_{a_0}, \sigma_{a_1}, \sigma_{b_1}, \sigma_{b_2}, \dots, \sigma_{b_{2g-1}}, \sigma_c, \tau_{e_1}, \tau_{e_2}, \dots, \tau_{e_{n-1}}\}$$

generates $\mathcal{M}(\Sigma_{g,0}, \mathcal{P}_n)$, and the mapping

$$\begin{aligned} x_i &\mapsto \sigma_{a_i} \quad (i = 0, 1), \quad y_i \mapsto \sigma_{b_i} \quad (1 \leq i \leq 2g-1), \\ z &\mapsto \sigma_c, \quad v_j \mapsto \tau_{e_j} \quad (1 \leq j \leq n-1), \end{aligned}$$

induces a homomorphism $\rho : G_{\Gamma(g,1,n)} \rightarrow \mathcal{M}(\Sigma_{g,0}, \mathcal{P}_n)$. Here again, the kernel of ρ was calculated in [124] for $n = 0$, and in [110] for $n \geq 1$.

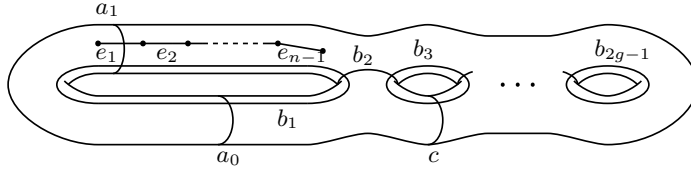


Figure 7.7. Generators of $\mathcal{M}(\Sigma_{g,0}, \mathcal{P}_n)$.

In order to state the results of [124] and [110], we need the following notations. Let Γ be a Coxeter graph, let $M = (m_{st})_{s,t \in S}$ be the Coxeter matrix of Γ , and let (G, Σ) be the Artin system of type Γ . For $X \subset S$, we denote by Γ_X the full subgraph of Γ generated by X , we set $\Sigma_X = \{\sigma_s; s \in X\}$, and we denote by G_X the subgroup of G generated by Σ_X . By [116], (G_X, Σ_X) is the Artin system of type Γ_X (see also [131]). If Γ_X is of spherical type, then we denote by $\Delta(X)$ the Garside element of (G_X, Σ_X) , viewed as an element of G .

Theorem 7.2 (Matsumoto [124]). (1) $\mathcal{M}(\Sigma_{g,1})$ is isomorphic with the quotient of $G_{\Gamma(g,1,0)}$ by the following relations

$$\begin{aligned} (R1) \quad & \Delta(y_1, y_2, y_3, z)^4 = \Delta(x_0, y_1, y_2, y_3, z)^2 \quad \text{if } g \geq 2 \\ (R2) \quad & \Delta(y_1, y_2, y_3, y_4, y_5, z)^2 = \Delta(x_0, y_1, y_2, y_3, y_4, y_5, z) \quad \text{if } g \geq 3 \end{aligned}$$

(2) $\mathcal{M}(\Sigma_{g,0})$ is isomorphic with the quotient of $G_{\Gamma(g,1,0)}$ by the above relations (R1) and (R2) together with

$$\begin{aligned} (R3) \quad & (x_0 y_1)^6 = 1 \quad \text{if } g = 1 \\ & x_0^{2g-2} = \Delta(y_2, y_3, z, y_4, \dots, y_{2g-1}) \quad \text{if } g \geq 2 \end{aligned}$$

Theorem 7.3 (Labruère, Paris [110]). Let $g \geq 1$, $r \geq 1$, and $n \geq 0$. Then $\mathcal{M}(\Sigma_{g,r}, \mathcal{P}_n)$ is isomorphic with the quotient of $G_{\Gamma(g,r,n)}$ by the following relations.

- *Relations from $\mathcal{M}(\Sigma_{g,1})$.*

$$(R1) \quad \Delta(y_1, y_2, y_3, z)^4 = \Delta(x_0, y_1, y_2, y_3, z)^2 \quad \text{if } g \geq 2$$

$$(R2) \quad \Delta(y_1, y_2, y_3, y_4, y_5, z)^2 = \Delta(x_0, y_1, y_2, y_3, y_4, y_5, z) \quad \text{if } g \geq 3$$

- *Relations of commutation.*

$$(R3) \quad \begin{aligned} & x_k \cdot \Delta(x_{i+1}, x_j, y_1)^{-1} x_i \Delta(x_{i+1}, x_j, y_1) \\ &= \Delta(x_{i+1}, x_j, y_1)^{-1} x_i \Delta(x_{i+1}, x_j, y_1) \cdot x_k \\ & \quad \text{if } 0 \leq k < j < i \leq r-1 \end{aligned}$$

$$(R4) \quad \begin{aligned} & y_2 \cdot \Delta(x_{i+1}, x_j, y_1)^{-1} x_i \Delta(x_{i+1}, x_j, y_1) \\ &= \Delta(x_{i+1}, x_j, y_1)^{-1} x_i \Delta(x_{i+1}, x_j, y_1) \cdot y_2 \\ & \quad \text{if } 0 \leq j < i \leq r-1 \text{ and } g \geq 2 \end{aligned}$$

- *Expressions of the u_i 's.*

$$(R5) \quad u_1 = \Delta(x_0, x_1, y_1, y_2, y_3, z) \cdot \Delta(x_1, y_1, y_2, y_3, z)^{-2} \quad \text{if } g \geq 2$$

$$(R6) \quad \begin{aligned} u_{i+1} &= \Delta(x_i, x_{i+1}, y_1, y_2, y_3, z) \cdot \Delta(x_{i+1}, y_1, y_2, y_3, z)^{-2} \\ & \quad \cdot \Delta(x_0, x_{i+1}, y_1)^2 \cdot \Delta(x_0, x_i, x_{i+1}, y_1)^{-1} \\ & \quad \text{if } 1 \leq i \leq r-2 \text{ and } g \geq 2 \end{aligned}$$

- *Other relations.*

$$(R7) \quad \Delta(x_{r-1}, x_r, y_1, v_1) = \Delta(x_r, y_1, v_1)^2 \quad \text{if } n \geq 2$$

$$(R8a) \quad \Delta(x_0, x_1, y_1, y_2, y_3, z) = \Delta(x_1, y_1, y_2, y_3, z)^2 \quad \text{if } n \geq 1, g \geq 2, \text{ and } r = 1$$

$$(R8b) \quad \begin{aligned} & \Delta(x_{r-1}, x_r, y_1, y_2, y_3, z) \cdot \Delta(x_r, y_1, y_2, y_3, z)^{-2} \\ &= \Delta(x_0, x_{r-1}, x_r, y_1) \cdot \Delta(x_0, x_r, y_1)^{-2} \\ & \quad \text{if } n \geq 1, g \geq 2, \text{ and } r \geq 2 \end{aligned}$$

Note that only the relations (R1), (R2), (R7), and (R8a) remain in the presentation if $r = 1$, and (R8a) must be replaced by (R8b) if $r \geq 2$. Note also that, if $g \geq 2$, then u_1, \dots, u_{r-1} can be removed from the generating set. However, to do so, one must add new long relations.

Theorem 7.4 (Labruère, Paris [110]). *Let $g \geq 1$ and $n \geq 1$. Then $\mathcal{M}(\Sigma_{g,0}, \mathcal{P}_n)$ is isomorphic with the quotient of $G_{\Gamma(g,1,n)}$ by the following relations.*

- *Relations from $\mathcal{M}(\Sigma_{g,1}, \mathcal{P}_n)$.*

$$(R1) \quad \Delta(y_1, y_2, y_3, z)^4 = \Delta(x_0, y_1, y_2, y_3, z)^2 \\ \text{if } g \geq 2$$

$$(R2) \quad \Delta(y_1, y_2, y_3, y_4, y_5, z)^2 = \Delta(x_0, y_1, y_2, y_3, y_4, y_5, z) \\ \text{if } g \geq 3$$

$$(R7) \quad \Delta(x_0, x_1, y_1, v_1) = \Delta(x_1, y_1, v_1)^2 \\ \text{if } n \geq 2$$

$$(R8a) \quad \Delta(x_0, x_1, y_1, y_2, y_3, z) = \Delta(x_1, y_1, y_2, y_3, z)^2 \\ \text{if } n \geq 1 \text{ and } g \geq 2$$

- *Other relations.*

$$(R9a) \quad x_0^{2g-n-2} \cdot \Delta(x_1, v_1, \dots, v_{n-1}) = \Delta(z, y_2, \dots, y_{2g-1})^2 \\ \text{if } g \geq 2$$

$$(R9b) \quad x_0^n = \Delta(x_1, v_1, \dots, v_{n-1}) \\ \text{if } g = 1$$

$$(R9c) \quad \Delta(x_0, y_1)^4 = \Delta(v_1, \dots, v_{n-1})^2 \\ \text{if } g = 1$$

Note. Presentations of $\mathcal{M}(\Sigma_{g,r})$, also in terms of Artin groups, with more generators but simpler relations, were obtained by Gervais in [88]. On the other hand, a unified proof of all these presentations can be found in [11].

7.3 Classification

This subsection is an account of Castel's results [40] on the geometric representations of the braid group \mathcal{B}_n on mapping class groups of surfaces of genus $g \leq \frac{n-1}{2}$.

Suppose first that n is odd, $n \geq 5$. Write $n = 2k + 1$, where $k \geq 2$. Let $r \geq 0$. We present the surface $\Sigma_{k,r}$ as the union of three subsurfaces, Ω_0 , \mathbb{A} , and Ω_1 , where Ω_0 is a surface of genus k with one boundary component, c , Ω_1 is a surface of genus 0 with $r + 1$ boundary components, c', d_1, \dots, d_r , and \mathbb{A} is an annulus bounded by c and c' (see Figure 7.8). Consider the essential circles a_1, a_2, \dots, a_{2k} drawn in Figure 7.8. Then, by Proposition 7.1, there exists a homomorphism $\rho_M : \mathcal{B}_n \rightarrow \mathcal{M}(\Sigma_{k,r})$ which sends σ_i to σ_{a_i} for all $1 \leq i \leq n - 1 = 2k$.

The statement of Castel's classification of the geometric representations of \mathcal{B}_n in $\mathcal{M}(\Sigma_{k,r})$ involves the centralizer of $\text{Im } \rho_M$ in $\mathcal{M}(\Sigma_{k,r})$. That is why we start with a description of the latter.

The inclusion of Ω_1 in $\Sigma_{k,r}$ induces a homomorphism $\mathcal{M}(\Omega_1) \rightarrow \mathcal{M}(\Sigma_{k,r})$ which is injective (see [133]). It is easily checked that the image of this homomorphism is contained in the centralizer of $\text{Im } \rho_M$. Another element of the

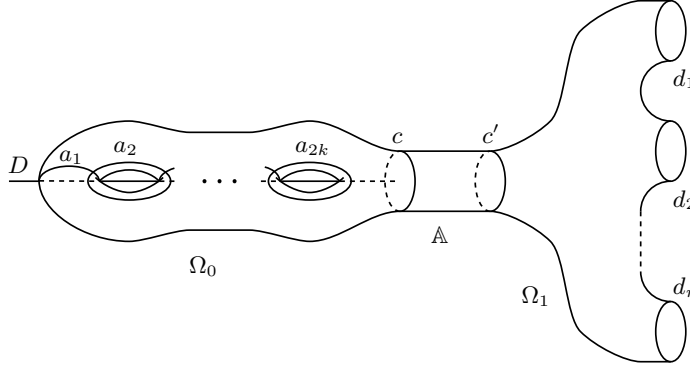


Figure 7.8. Decomposition of $\Sigma_{k,r}$ (n odd).

centralizer is the element $u \in \mathcal{M}(\Sigma_{k,r})$ represented by the homeomorphism $U : \Sigma_{k,r} \rightarrow \Sigma_{k,r}$ which is the axial symmetry relative to the axis D on Ω_0 , a half-twist which pointwise fixes c' on the annulus \mathbb{A} , and the identity on Ω_1 .

Proposition 7.5 (Castel [40]). *The centralizer of $\text{Im } \rho_M$ in $\mathcal{M}(\Sigma_{k,r})$ is generated by $\mathcal{M}(\Omega_1) \cup \{u\}$.*

If $r = 0$, then $\mathcal{M}(\Omega_1) = \{1\}$, u is of order 2, and $Z_{\mathcal{M}(\Sigma_{k,r})}(\text{Im } \rho_M) = \langle u \rangle$ is cyclic of order 2. If $r = 1$, then $\mathcal{M}(\Omega_1) = \langle \tau_c \rangle$, $u^2 = \tau_c$, and $Z_{\mathcal{M}(\Sigma_{k,r})}(\text{Im } \rho_M) = \langle u \rangle$ is an infinite cyclic group. If $r = 2$, then $Z_{\mathcal{M}(\Sigma_{k,r})}(\text{Im } \rho_M)$ is a free abelian group of rank 3 freely generated by $\{u, \sigma_{d_1}, \sigma_{d_2}\}$. If $r \geq 3$, then $Z_{\mathcal{M}(\Sigma_{k,r})}(\text{Im } \rho_M)$ is more complicated.

For $\varepsilon \in \{\pm 1\}$ and $z \in Z_{\mathcal{M}(\Sigma_{k,r})}(\text{Im } \rho_M)$, the mapping $\sigma_i \mapsto \sigma_{a_i}^\varepsilon z$, $1 \leq i \leq n-1$, induces a homomorphism $\rho_M(\varepsilon, z) : \mathcal{B}_n \rightarrow \mathcal{M}(\Sigma_{k,r})$ called the *transvection of ρ_M by (ε, z)* . On the other hand, a homomorphism $\varphi : \mathcal{B}_n \rightarrow G$, where G is a group, is called *cyclic* if there exists $\alpha \in G$ such that $\varphi(\sigma_i) = \alpha$ for all $1 \leq i \leq n-1$.

Theorem 7.6 (Castel [40]). *Suppose n odd, $n \geq 5$, and set $n = 2k + 1$. Let $g \geq 0$ and $r \geq 0$.*

- (1) *If $g < k$, then all the homomorphisms $\varphi : \mathcal{B}_n \rightarrow \mathcal{M}(\Sigma_{g,r})$ are cyclic.*
- (2) *All the non-cyclic homomorphisms $\varphi : \mathcal{B}_n \rightarrow \mathcal{M}(\Sigma_{k,r})$ are conjugate to transvections of ρ_M .*
- (3) *The homomorphism $\rho_M : \mathcal{B}_n \rightarrow \mathcal{M}(\Sigma_{k,r})$ is injective if and only if $r \geq 1$.*

Now, we suppose that n is even, $n \geq 6$, and we set $n = 2k + 2$. We choose $r_1, r_2 \geq 0$ such that $r_1 + r_2 = r$ and we represent the surface $\Sigma_{k,r}$

as the union of three subsurfaces, a surface Ω_0 of genus k with two boundary components, c_1 and c_2 , a surface Ω_1 of genus 0 with $r_1 + 1$ boundary components c_1, d_1, \dots, d_{r_1} , and a surface Ω_2 of genus 0 with $r_2 + 1$ boundary components $c_2, d_{r_1+1}, \dots, d_{r_1+r_2}$ (see Figure 7.9). Consider the essential circles a_1, \dots, a_{n-1} drawn in Figure 7.9. Then, by Proposition 7.1, there exists a homomorphism $\rho_M(r_1, r_2) : \mathcal{B}_n \rightarrow \mathcal{M}(\Sigma_{k,r})$ which sends σ_i to σ_{a_i} for all $1 \leq i \leq n-1$.

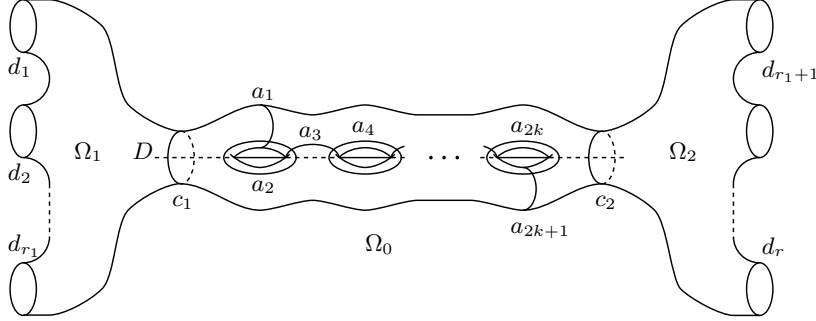


Figure 7.9. Decomposition of $\Sigma_{k,r}$ (n even).

The inclusions $\Omega_1, \Omega_2 \subset \Sigma_{k,r}$ induce a homomorphism $\mathcal{M}(\Omega_1) \times \mathcal{M}(\Omega_2) \rightarrow \mathcal{M}(\Sigma_{k,r})$ which is injective (see [133]), and we have:

Proposition 7.7 (Castel [40]). (1) *If $r > 0$, then the centralizer of $\text{Im } \rho_M(r_1, r_2)$ in $\mathcal{M}(\Sigma_{k,r})$ is $\mathcal{M}(\Omega_1) \times \mathcal{M}(\Omega_2)$.*
(2) *If $r = 0$, then the centralizer of $\text{Im } \rho_M(r_1, r_2)$ in $\mathcal{M}(\Sigma_{k,r})$ is a cyclic group of order 2 generated by an element represented by the axial symmetry relative to the axis D of Figure 7.9.*

For $\varepsilon \in \{\pm 1\}$ and $z \in Z_{\mathcal{M}(\Sigma_{k,r})}(\text{Im } \rho_M(r_1, r_2))$, the mapping $\sigma_i \mapsto \sigma_{a_i}^\varepsilon z$, $1 \leq i \leq n-1$, induces a homomorphism $\rho_M(r_1, r_2, \varepsilon, z) : \mathcal{B}_n \rightarrow \mathcal{M}(\Sigma_{k,r})$ called the *transvection* of $\rho_M(r_1, r_2)$ by (ε, z) .

Theorem 7.8 (Castel [40]). *Suppose n even, $n \geq 6$, and set $n = 2k + 2$. Let $g \geq 0$ and $r \geq 0$.*

- (1) *If $g < k$, then all the homomorphisms $\varphi : \mathcal{B}_n \rightarrow \mathcal{M}(\Sigma_{g,r})$ are cyclic.*
- (2) *If $\varphi : \mathcal{B}_n \rightarrow \mathcal{M}(\Sigma_{k,r})$ is a non-cyclic homomorphism, then there exist $r_1, r_2 \geq 0$ such that $r_1 + r_2 = r$ and φ is conjugate to a transvection of $\rho_M(r_1, r_2)$.*
- (3) *Let $r_1, r_2 \geq 0$ such that $r_1 + r_2 = r$. The homomorphism $\rho_M(r_1, r_2) : \mathcal{B}_n \rightarrow \mathcal{M}(\Sigma_{k,r})$ is injective if and only if $r_1 \geq 1$ and $r_2 \geq 1$.*

Recall that, for a group G , $\text{Out}(G)$ denotes the group of outer automorphisms of G . Now, Theorems 7.6 and 7.8 can be used for new proofs of the following two theorems.

Theorem 7.9 (Dyer, Grossman [77]). *We have $\text{Out}(\mathcal{B}_n) = \mathbb{Z}/2\mathbb{Z}$ if $n \geq 5$.*

Theorem 7.10 (Ivanov [98], McCarthy [125]). *Let $g \geq 2$ and $r \geq 0$. Then*

$$\text{Out}(\mathcal{M}(\Sigma_{g,r})) = \begin{cases} \{1\} & \text{if } r \geq 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } r = 0 \text{ and } g \geq 3 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } r = 0 \text{ and } g = 2 \end{cases}$$

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